

# On Asymptotic Expansions in Spin Boson Models

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## Abstract

We consider expansions of eigenvalues and eigenvectors of models of quantum field theory. For a class of models known as generalized spin boson model we prove the existence of asymptotic expansions of the ground state and the ground state energy to arbitrary order. We need a mild but very natural infrared assumption, which is weaker than the assumption usually needed for other methods such as operator theoretic renormalization to be applicable. The result complements previously shown analyticity properties.

## 1 Introduction

Perturbation theory is widely used to calculate various quantities in quantum mechanics. As long as the perturbation is “small” compared to the unperturbed system one expects to obtain good approximations to physical quantities. In particular, in case of isolated eigenvalues analytic perturbation theory is available allowing the calculation of eigenvalues and eigenvectors in terms of convergent power series, which are also known as Rayleigh-Schrödinger perturbation series [20, 19]. However, in many-body quantum systems and models of massless quantum fields the ground state is typically not isolated from the rest of the spectrum and analytic perturbation theory is not applicable. Different methods to cope with these problems have been developed [20, 13, 2].

In this paper we consider models of massless quantum fields. Specifically, we consider a quantum mechanical system with finitely many degrees of freedom, which is linearly coupled to a field of relativistic massless bosons. Such models are also known as generalized spin-boson models. They are used to describe low energy aspects of non-relativistic quantum mechanical matter interacting with a quantized radiation field such as a field of phonons or a field of photons. Various spectral properties of the Hamiltonians of such models have been investigated. In particular, we assume that the quantum field is massless. This implies that the ground state energy as well as resonance energies are not isolated from the rest of the spectrum. Existence of ground states and resonance states have been shown to exist for such models [11, 5, 6, 7, 12, 14, 2]. In spite that the ground state energy is embedded in continuous spectrum

and analytic perturbation theory is not applicable, it has been shown in various situations that the ground state and the ground state energy are in fact analytic functions of the coupling constant [13, 18, 1, 2]. To prove these results one uses operator theoretic renormalization [4] and in some cases one can employ expansion techniques from statistical mechanics. The analyticity results obtained by renormalization are rather surprising. The calculation of the Rayleigh-Schrödinger expansion coefficients involve sums of divergent expressions, and it is at first sight not obvious in which situations these infinities will eventually cancel each other. On the other hand there exist situations where the ground state energy is not an analytic function of the coupling constant [9].

In this paper we show that for a large class of generalized spin boson models there exist asymptotic expansions for the ground state and the ground state energy to arbitrary order. Whereas the existence of asymptotic expansion is weaker than the existence of an analytic expansion, our result holds in situations where analytic expansions have not been shown. We expect that our technique can be used to derive asymptotic expansions in situations where analytic expansions in fact do not exist. Such a situation may occur when the unperturbed operator has a degenerate ground state energy, which is lifted once the interaction is turned on. This will be addressed in a forthcoming paper by the authors.

We want to mention that for models which we consider asymptotic expansions have been investigated in several papers. In particular expansions of the first few orders have been investigated in [15, 10, 8, 9]. More recently in [3] asymptotic expansion formulas have been studied to arbitrary order, provided the infrared regularization is sufficiently strong, i.e., the higher the order of expansion the stronger the infrared regularization. In the present paper we relax this infrared assumption substantially. Our main result of the paper, Theorem 2, stated below, shows the existence of an asymptotic expansion for a reasonable infrared assumption. The key idea in the proof is to show that the infinities involved in calculating the Rayleigh-Schrödinger expansion coefficients cancel out. Showing that these cancellations can be controlled to arbitrary order, without any analyticity assumption, is the main new technical contribution of the present paper.

The paper is organized as follows. In the next section we introduce the model and state the main result. In Section 3 we derive for a general class of models formulas for expansion coefficients of the ground state and the ground state energy in terms of the coupling constant. Assuming that the expansion coefficients are finite, which will be shown in Sections 4 and 5, we determine general conditions for which these expansion coefficients yield an asymptotic expansion.

In Section 4 we show Theorem 1, i.e., the finiteness of the expansion coefficients of the ground state energy. To this end, we first express the expansion coefficients as a sum of linked contractions involving renormalized propagators, which we call renormalized Feynman graphs. The renormalized propagators take into account the cancellations which results in an improved infrared behaviour. Finally we estimate the renormalized Feynman graphs and prove the finiteness of each expansion coefficient.

Assuming a certain condition we show in Section 5 the finiteness of the expansion coefficients for the ground state. Similarly to Section 4 we first express the squared of the norm of the expansion coefficients as a sum of linked contractions involving renormalized propagators, except the one in the middle. We then use that formula to show the finiteness of the expansion coefficients of the ground state.

In Section 6 we collect the results of the previous sections and provide a proof of Theorem 2.

## 2 Model and Statement of Main Results

In this section we introduce the model and state the main result. Let  $\mathcal{H}_{\text{at}}$  be a separable Hilbert space and let  $H_{\text{at}}$  be a selfadjoint operator in  $\mathcal{H}_{\text{at}}$ . Assume that  $E_{\text{at}} = \inf \sigma(H_{\text{at}})$  is a nondegenerate eigenvalue of  $H_{\text{at}}$ , which is isolated from the rest of the spectrum, i.e.,

$$E_{\text{at}} < \epsilon_1 := \inf(\sigma(H_{\text{at}}) \setminus \{E_1\}).$$

Let  $\varphi_{\text{at}}$  denote the eigenvector and let  $P_{\text{at}}$  denote the orthogonal eigenprojection of  $E_{\text{at}}$ . For a separable Hilbert space  $\mathfrak{h}$  we write

$$L_s^2((\mathbb{R}^3)^n; \mathfrak{h}) := \left\{ \psi \in L^2((\mathbb{R}^3)^n; \mathfrak{h}) : \psi(k_1, \dots, k_n) = \psi(k_{\pi(1)}, \dots, k_{\pi(n)}) \right. \\ \left. \forall \text{ permutations } \pi \text{ of } \{1, \dots, n\} \right\}.$$

We introduce the symmetric Fock space

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{F}_n,$$

where the so called  $n$ -photon subspaces are defined by

$$\mathcal{F}_0 := \mathbb{C}, \\ \mathcal{F}_n := L_s^2((\mathbb{R}^3)^n; \mathbb{C}).$$

We introduce the so called vacuum vector  $\Omega = (1, 0, 0, \dots) \in \mathcal{F}$ . The free field Hamiltonian is defined by

$$H_f : \text{dom}(H_f) \subset \mathcal{F} \rightarrow \mathcal{F} \\ (H_f \psi)_n(k_1, \dots, k_n) := (|k_1| + |k_2| + \dots + |k_n|) \psi_n(k_1, \dots, k_n),$$

where  $\text{dom}(H_f) := \{\psi \in \mathcal{F} : H_f \psi \in \mathcal{F}\}$ . The total Hilbert space is defined by

$$\mathcal{H} := \mathcal{H}_{\text{at}} \otimes \mathcal{F} \simeq \bigoplus_{n=0}^{\infty} L_s^2((\mathbb{R}^3)^n; \mathcal{H}_{\text{at}}).$$

We shall identify the spaces on the right hand side and occasionally drop the tensor sign in the notation. For  $G : \mathbb{R}^3 \rightarrow \mathcal{L}(\mathcal{H}_{\text{at}})$  a strongly measurable function such that

$$\int \|G(k)\|^2 dk < \infty,$$

we define the so called annihilation operator

$$a(G) : \text{dom}(a(G)) \subset \mathcal{H} \rightarrow \mathcal{H} \\ \psi \mapsto (a(G)\psi)_n(k_1, \dots, k_n) := \sqrt{n+1} \int G^*(k) \psi_{n+1}(k, k_1, \dots, k_n) dk,$$

where  $\text{dom}(a(G)) := \{\psi \in \mathcal{H} : a(G)\psi \in \mathcal{H}\}$ . One readily verifies that  $a(G)$  is a densely defined closed operator. We denote its adjoint by  $a^*(G) := a(G)^*$ , and introduce the field operator by

$$\phi(G) := \overline{a(G) + a^*(G)},$$

where the line denotes the closure.

To define the total Hamiltonian we assume in addition that

$$\int \|G(k)\|^2 (1 + |k|^{-1}) dk < \infty, \quad (2.1)$$

since then it is well known that  $\phi(G)$  is infinitesimally small with respect to  $1_{\mathcal{H}_{\text{at}}} \otimes H_f$ . This allows us to define the total Hamiltonian of the interacting system by

$$H(\lambda) = H_{\text{at}} \otimes 1_{\mathcal{F}} + 1_{\mathcal{H}_{\text{at}}} \otimes H_f + \lambda V, \quad (2.2)$$

where  $\lambda \in \mathbb{R}$  is the coupling constant and  $V = \phi(G)$ , as a selfadjoint operator on the domain  $\text{dom}(H(0))$ . Let

$$E(\lambda) = \inf \sigma(H(\lambda)).$$

Below we shall make the following assumption

**Hypothesis 1.** There exists a positive constant  $\lambda_0$  such that for all  $\lambda \in [0, \lambda_0]$  the number  $E(\lambda)$  is a simple eigenvalue of  $H(\lambda)$  with eigenvector  $\psi(\lambda) \in \mathcal{H}$ .

*Remark 1.* We note that the existence of ground states has been verified in many cases [11, 21, 7, 12, 14]. In particular, it has been shown in [12] that Hypothesis 1 holds if  $H_{\text{at}}$  has compact resolvent and the coupling function satisfies

$$\int \|G(k)\|^2 (1 + |k|^{-2}) dk < \infty. \quad (2.3)$$

We will outline in the next section, that if one formally expands the eigenvalue equation for the ground state in powers of the coupling constant  $\lambda$  and inductively solves for the expansion coefficients of the ground state energy one obtains the recursion relation (2.8), below. One can show that these expansion coefficients are indeed finite, which is the content of the next theorem. To formulate it we introduce the following notations. We write

$$H_0 = H(0)$$

and

$$\psi_0 = \varphi_{\text{at}} \otimes \Omega,$$

and denote by  $P_0$  the projection onto  $\psi_0$  and let  $\bar{P}_0 = 1 - P_0$ . Let  $P_\Omega$  denote the orthogonal projection in  $\mathcal{F}$  onto  $\Omega$ . Then we can write

$$\bar{P}_0 = P_{\text{at}} \otimes \bar{P}_\Omega + \bar{P}_{\text{at}} \otimes 1_{\mathcal{F}}, \quad (2.4)$$

where  $\bar{P}_\Omega = 1_{\mathcal{F}} - P_\Omega$  and  $\bar{P}_{\text{at}} = 1_{\mathcal{H}_{\text{at}}} - P_{\text{at}}$ .

**Theorem 1.** Suppose that (2.3) holds. Then there exists a unique sequence  $(E_n)_{n \in \mathbb{N}}$  in  $\mathbb{R}$  such that

$$E_0 = E_{\text{at}} \quad (2.5)$$

$$E_1 = \langle \psi_0, V \psi_0 \rangle \quad (2.6)$$

$$E_n = \lim_{\eta \downarrow 0} E_n(\eta), \quad n \geq 2, \quad (2.7)$$

where

$$E_n(\eta) \quad (2.8)$$

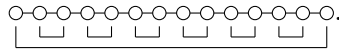
$$:= \sum_{k=2}^n \sum_{\substack{j_1 + \dots + j_k = n \\ j_s \geq 1}} \langle \psi_0, (\delta_{1j_1} V - E_{j_1}) \prod_{s=2}^k \{ (E_0 - \eta - H_0)^{-1} \bar{P}_0 (\delta_{1j_s} V - E_{j_s}) \} \psi_0 \rangle$$

In particular the limit on the right hand side of (2.8) exists and is a finite number. The sequence  $(E_n)_{n \in \mathbb{N}}$  can be defined inductively using (2.5)–(2.7).

*Remark 2.* We note that the positive number  $\eta$  appearing in (2.8) serves as a regularization. The theorem states that the limit exists as the regularization is removed. We note that this is not obvious, as some of the individual terms on the right hand side of (2.8) diverge. This can be illustrated as follows. Consider for  $n = 2m$  the summand where  $j_s = 1$  for all  $s$ . Inserting (2.4) and  $a^*(G) + a(G)$  for  $V$ , multiplying out the resulting expression, using Wicks theorem and the so called pull through formula [5, Appendix A] one obtains various terms. One of them being

$$(-1)^{n-1} \int dk_1 \dots dk_m \langle \varphi_{\text{at}}, G^*(k_1) \frac{P_{\text{at}}}{|k_1| + \eta} G^*(k_2) \frac{P_{\text{at}}}{|k_1| + |k_2| + \eta} G(k_2) \frac{P_{\text{at}}}{|k_1| + \eta} \dots G^*(k_m) \frac{P_{\text{at}}}{|k_1| + |k_m| + \eta} G(k_m) \frac{P_{\text{at}}}{|k_1| + \eta} G(k_1) \varphi_{\text{at}} \rangle,$$

which is obtained by contracting the first and the last entry of the interaction and contracting the remaining nearest neighbor pairs. This can be symbolically pictured as follows



If  $\eta \downarrow 0$  the integral over  $k_1$  may become divergent for large  $m$ . This is the case, for example, if  $\int dk |k|^{-m} \|P_{\text{at}} G(k) P_{\text{at}}\|^2$  diverges for  $m$  sufficiently large. The convergence of (2.8) can be restored using cancellations originating from energy subtractions. To illustrate this, we note that for every nearest neighbor contraction there is a corresponding counter term  $E_2$ , i.e., we have

$$\begin{aligned} & \int dk_2 P_{\text{at}} G^*(k_2) \frac{1}{|k_1| + |k_2| + \eta} G(k_2) P_{\text{at}} - E_2 P_{\text{at}} \\ &= \int dk_2 P_{\text{at}} G^*(k_2) \left( \frac{1}{|k_1| + |k_2| + \eta} - \frac{1}{|k_2|} \right) G(k_2) P_{\text{at}} \\ &= -(|k_1| + \eta) \int dk_2 P_{\text{at}} G^*(k_2) \frac{1}{(|k_1| + |k_2| + \eta)|k_2|} G(k_2) P_{\text{at}} \end{aligned}$$

which remedies the singularity in  $k_1$  at the origin. To prove Theorem 1 we will show that this cancellation can be carried out at every order.

Once one has established the finiteness of the expansion coefficients of the ground state energy, we will show that this yields an asymptotic expansion of the ground state energy. This is the content of the following theorem.

**Theorem 2.** *Suppose (2.3) and Hypothesis 1 holds. Then the sequence  $(E_n)_{n \in \mathbb{N}}$  defined in Theorem 1 yields an asymptotic expansion of the ground state energy, i.e.,*

$$\lim_{\lambda \downarrow 0} \lambda^{-n} \left( E(\lambda) - \sum_{k=0}^n E_k \lambda^k \right) = 0.$$

*Remark 3.* We note that if we would have the infrared condition  $\int \|G(k)\|^2 (1+|k|^{-2-\mu}) dk < \infty$ , for some  $\mu > 0$ , which is slightly stronger than (2.3), then it would follow from [13] that one has analyticity. Moreover, there are couplings with (2.3) where additional symmetries may cancel infrared divergencies such that the ground state energy is analytic [16, 17].

*Remark 4.* Note that in view of Remark 1 Hypothesis 1 is not a restrictive assumption. And in many situations follows already from Inequality (2.3).

In the remaining parts of the paper we provide proofs of the above results and furthermore we also show the finiteness of the expansion coefficients for the ground state.

### 3 Asymptotic Perturbation Theory

In this section we derive formulas for the expansion coefficients of the ground state and its energy. Moreover we show that provided these coefficients are finite up to some order, say  $n$ , and a continuity assumption for the ground state holds, then the ground state energy has an asymptotic expansion up to order  $n$ . We shall derive this result with two different methods. The first method in Subsection 3.1 uses formal expansions and the comparison of coefficients combined with an analytic estimate. The second method outlined in Subsection 3.2 is based on a Feshbach type argument together with a resolvent expansion.

We state our result for more general operators than introduced in the previous section. Nevertheless we will use the same symbols as in the previous section. Let  $V$  and  $H_0$  be selfadjoint operators in a Hilbert space  $\mathcal{H}$ . To prove our results we will use the following assumption.

**Hypothesis 2.** The operator  $H_0$  is bounded from below and  $V$  is  $H_0$ -bounded. There exists a positive constant  $\lambda_0$  such that for all  $\lambda \in [0, \lambda_0]$  there exists a simple eigenvalue  $E(\lambda)$  of

$$H(\lambda) = H_0 + \lambda V$$

with eigenvector  $\psi(\lambda)$ . Moreover,

$$\lim_{\lambda \rightarrow 0} \psi(\lambda) = \psi(0) \neq 0, \quad \lim_{\lambda \rightarrow 0} E(\lambda) = E(0) \tag{H}$$

and

$$\langle \psi(0), \psi(\lambda) \rangle = 1 \tag{N}$$

for all  $\lambda \in [0, \lambda_0]$ .

We note that (N) can always be achieved using a suitable normalization, possibly making the positive number  $\lambda_0$  smaller. For notational convenience we shall write

$$E_0 = E(0), \quad \psi_0 = \psi(0).$$

Let  $P_0$  denote the projection onto the kernel of  $H_0 - E_0$  and let  $\bar{P}_0 = 1 - P_0$ .

### 3.1 Expansion Method

The idea behind the expansion method is to expand the eigenvalue equation in a formal power series and equating coefficients. This will lead to Eq. (3.1). In Lemma 1 we show that provided one has a solution of (3.1) up to some order  $n$ , then the ground state energy has an asymptotic expansion up to the same order, provided Hypothesis 2 holds. In Lemma 2 we inductively solve (3.1), and in Lemma 3 we give an explicit formula for the inductive solution. We note that a similar result has been obtained in [3]. However in contrast to the result in [3] we have less restrictive assumptions.

**Lemma 1.** *Suppose Hypothesis 2 holds. Let  $n \in \mathbb{N}$  and suppose there exist  $\psi_1, \dots, \psi_n \in \bar{P}_0 \mathcal{H}$  and  $E_1, \dots, E_n \in \mathbb{C}$  such that for all  $m \in \mathbb{N}$  with  $m \leq n$  we have*

$$H_0 \psi_m + V \psi_{m-1} = \sum_{k=0}^m E_k \psi_{m-k}. \quad (3.1)$$

Then for all  $m \in \{1, \dots, n\}$  we have that

$$\lim_{\lambda \downarrow 0} \lambda^{-m} \left( E(\lambda) - \sum_{k=0}^m E_k \lambda^k \right) = 0, \quad (3.2)$$

$$\lim_{\lambda \downarrow 0} \lambda^{-m} \langle \psi_0, V(\psi(\lambda) - \sum_{k=0}^m \psi_k \lambda^k) \rangle = 0. \quad (3.3)$$

First observe that (3.1) implies that for all  $m \leq n$  we have

$$\langle \psi_0, V \psi_{m-1} \rangle = E_m.$$

*Proof.* Proof by induction in  $n$ . We define for  $\lambda \in (0, \lambda_0)$  the quantities

$$e_n(\lambda) := \lambda^{-n} (E(\lambda) - (E_0 + \lambda E_1 + \lambda^2 E_2 + \dots + \lambda^n E_n))$$

$$\rho_n(\lambda) := \lambda^{-n} (\psi(\lambda) - (\psi_0 + \lambda \psi_1 + \lambda^2 \psi_2 + \dots + \lambda^n \psi_n)).$$

Equation (3.3) for  $m = 0$  is just Hypothesis 2. Thus it remains to show the induction step.

The eigenvalue equation gives

$$\bar{P}_0 (H(\lambda) - E(\lambda)) \bar{P}_0 \psi(\lambda) = -\bar{P}_0 V P_0 \psi(\lambda). \quad (3.4)$$

$n - 1 \rightarrow n$ : Suppose that (3.1) holds for all  $m \in \{1, \dots, n\}$ . By induction Hypothesis we know that  $\lambda E_n + \lambda e_n(\lambda) \rightarrow 0$  and  $\langle V \psi_0, \lambda \psi_n + \lambda \rho_n(\lambda) \rangle \rightarrow 0$ . From the eigenvalue equation we find

$$(H_0 + \lambda V) \left[ \sum_{k=0}^n \lambda^k \psi_k + \lambda^n \rho_n(\lambda) \right] = \left( \sum_{k=0}^n \lambda^k E_k + \lambda^n e_n(\lambda) \right) \left[ \sum_{k=0}^n \lambda^k \psi_k + \lambda^n \rho_n(\lambda) \right].$$

By ordering according to powers of  $\lambda$  we see from (3.1) that many terms vanish and

$$\lambda V\psi_n + (H_0 + \lambda V)\rho_n(\lambda) = \rho_n(\lambda)E(\lambda) + e_n(\lambda) \sum_{k=0}^n \lambda^k \psi_k + \sum_{k=n+1}^{2n} \lambda^{k-n} \sum_{j=k-n}^n E_j \psi_{k-j}. \quad (3.5)$$

If one applies  $P_0$  to equation (3.5) one obtains

$$\lambda P_0 V(\psi_n + \rho_n(\lambda)) = e_n(\lambda).$$

By induction Hypothesis the left hand side tends to zero as  $\lambda \rightarrow 0$ . This shows that (3.2) holds for all  $m \in \{1, \dots, n\}$ . Solving for terms involving  $\rho_n(\lambda)$  in (3.5) we arrive at

$$(H(\lambda) - E(\lambda))\rho_n(\lambda) = e_n(\lambda) \sum_{k=0}^n \lambda^k \psi_k + \sum_{k=n+1}^{2n} \lambda^{k-n} \sum_{j=k-n}^n E_j \psi_{k-j} - \lambda V\psi_n.$$

Applying  $\bar{P}_0$  to this equation and using that  $P_0\rho_n(\lambda) = 0$  we find

$$\bar{P}_0(H(\lambda) - E(\lambda))\bar{P}_0\rho_n(\lambda) = \bar{P}_0 \left( e_n(\lambda) \sum_{k=0}^n \lambda^k \psi_k + \sum_{k=n+1}^{2n} \lambda^{k-n} \sum_{j=k-n}^n E_j \psi_{k-j} - \lambda V\psi_n \right).$$

Calculating the inner product with  $\psi(\lambda)$  and using (3.4) we find

$$\langle \psi(\lambda), P_0 V \rho_n(\lambda) \rangle = -\langle \bar{P}_0 \psi(\lambda), e_n(\lambda) \sum_{k=1}^n \lambda^{k-1} \psi_k + \sum_{k=n+1}^{2n} \lambda^{k-n-1} \sum_{j=k-n}^n E_j \psi_{k-j} - \bar{P}_0 V \psi_n \rangle.$$

This and Hypothesis 2 imply that (3.3) holds for all  $m \in \{1, \dots, n\}$ .  $\square$

Next we inductively solve Equation (3.1).

**Lemma 2.** (*Inductive Formula*) Let  $n \in \mathbb{N}$  and suppose there exist  $\psi_1, \dots, \psi_n \in \bar{P}_0 \mathcal{H}$  and  $E_1, \dots, E_n \in \mathbb{C}$  such that for all  $m \in \mathbb{N}$  with  $m \leq n$  we have

$$H_0 \psi_m + V \psi_{m-1} = \sum_{k=0}^m E_k \psi_{m-k}. \quad (3.6)$$

Then defining

$$E_{n+1} := \langle \psi_0, V \psi_n \rangle \quad (3.7)$$

as well as

$$\psi_{n+1} := (H_0 - E_0)^{-1} \bar{P}_0 \left( \sum_{k=1}^{n+1} E_k \psi_{n+1-k} - V \psi_n \right), \quad (3.8)$$

provided

$$\bar{P}_0 \left( \sum_{k=0}^n E_k \psi_{n-k} - V \psi_n \right) \in \text{dom} \left( (H_0 - E_0)^{-1} \bar{P}_0 \right), \quad (3.9)$$

we obtain a solution of (3.6) for  $m = n + 1$ .



We note that the assumption in (3.9) is less restrictive than the one in [3], which will turn out to be crucial to obtain the asymptotic expansion of the ground state to arbitrary order.

*Proof.* This follows by insertion of (3.8) and (3.7) into (3.6) for  $m = n + 1$ .  $\square$

If we solve the recursive relation of the previous lemma, we obtain the following formulas.

**Lemma 3.** (*Direct Formula*) Let  $n \in \mathbb{N}$  and suppose there exist  $\psi_1, \dots, \psi_n \in \bar{P}_0 \mathcal{H}$  and  $E_1, \dots, E_n \in \mathbb{C}$  such that the following holds. We have  $E_1 = \langle \psi_0, V \psi_0 \rangle$ , for all  $m \in \mathbb{N}$  with  $2 \leq m \leq n$  we have

$$E_m = - \sum_{k=2}^m \sum_{\substack{j_1 + \dots + j_k = m \\ j_s \geq 1}} \langle \psi_0, (E_{j_1} - \delta_{1j_1} V) \prod_{s=2}^k \{ (H_0 - E_0)^{-1} \bar{P}_0 (E_{j_s} - \delta_{1j_s} V) \} \psi_0 \rangle, \quad (3.10)$$

and for all  $m \in \mathbb{N}$  with  $m \leq n$  we have

$$\psi_m = \sum_{k=1}^m \sum_{\substack{j_1 + \dots + j_k = m \\ j_s \geq 1}} \prod_{s=1}^k \{ (H_0 - E_0)^{-1} \bar{P}_0 (E_{j_s} - \delta_{1j_s} V) \} \psi_0, \quad (3.11)$$

assuming that the expressions on the right hand side of (3.10) and (3.11) exist in the sense of Lemma 2. Then for all  $m \in \mathbb{N}$  with  $m \leq n$  we have

$$H_0 \psi_m + V \psi_{m-1} = \sum_{k=0}^m E_k \psi_{m-k}.$$

*Proof.* We prove this lemma by induction in  $n$ . The case  $n = 1$  follows from a straight forward calculation. Suppose the claim holds for  $n$ . Then also the assumption of Lemma 2 holds. Thus we define  $E_{n+1}$  as in (3.7)

$$\begin{aligned} E_{n+1} &:= \langle \psi_0, V \psi_n \rangle \\ &= - \sum_{k=2}^{n+1} \sum_{\substack{j_1 + \dots + j_k = n+1 \\ j_s \geq 1}} \langle \psi_0, (E_{j_1} - \delta_{1j_1} V) \prod_{s=2}^k \{ (H_0 - E_0)^{-1} \bar{P}_0 (E_{j_s} - \delta_{1j_s} V) \} \psi_0 \rangle, \end{aligned}$$

where in the second line we used the assumption (3.11) (note that  $\langle \psi_0, E_j \bar{P}_0(\cdot) \rangle = 0$ ), and we define  $\psi_{n+1}$  as in (3.8)

$$\begin{aligned} \psi_{n+1} &:= (H_0 - E_0)^{-1} \bar{P}_0 \left( \sum_{j=1}^{n+1} (E_j - \delta_{1j} V) \psi_{n+1-j} \right) \\ &= \sum_{k=1}^{n+1} \sum_{\substack{j_1 + \dots + j_k = n+1 \\ j_s \geq 1}} \prod_{s=1}^k \{ (H_0 - E_0)^{-1} \bar{P}_0 (E_{j_s} - \delta_{1j_s} V) \} \psi_0, \end{aligned}$$

where we wrote the first line with slightly different notation than in (3.8) and in the second line we used the assumption (3.11). Now it follows from Lemma 2 that the claim of the lemma holds also for  $n + 1$ .  $\square$

### 3.2 Resolvent Method

Here we use a Feshbach type or Schur complement argument together with a resolvent expansion. The proof of the Lemma in this subsection is inspired by [3].

**Lemma 4.** *Suppose that Hypothesis 2 holds. Assume that starting with  $K_0 := \frac{\bar{P}_0}{H_0 - E_0}$  and  $E_1 := \langle \psi_0, V\psi_0 \rangle$ , we can define recursively for  $m \in \{1, \dots, n-2\}$*

$$\begin{aligned} K_m &:= \sum_{j=1}^m K_{j-1}(E_{m+1-j} - \delta_{jm}V)K_0, \\ E_{m+1} &:= -\langle \psi_0, VK_{m-1}V\psi_0 \rangle, \end{aligned} \tag{3.12}$$

such that  $\bar{P}_0V\psi_0 \in \text{dom}(K_l)$  for  $l = 0, \dots, n-2$ . Then  $E(\lambda)$  has an asymptotic expansion up to order  $n$ , i.e., for all  $m = 1, \dots, n$

$$\lim_{\lambda \downarrow 0} \lambda^{-m} \left( E(\lambda) - \sum_{k=0}^m E_k \lambda^k \right) = 0.$$

*Remark 5.* The statement of Lemma 4 is equivalent to the statements of Lemma 3 and Lemma 1 combined. In particular, we may solve iteratively for  $K_m$  and obtain the relation

$$VK_{m-2}V = \sum_{k=2}^m \sum_{\substack{j_1 + \dots + j_k = m \\ j_s \geq 1}} (E_{j_s} - \delta_{1j_s}V) \prod_{s=2}^k \{ (H_0 - E_0)^{-1} \bar{P}_0 (E_{j_s} - \delta_{1j_s}V) \}.$$

Moreover, given  $E(\lambda)$ , we can recover  $\psi(\lambda)$  by

$$\psi(\lambda) = \psi_0 - \lambda \bar{P}_0 [\bar{P}_0 (H(\lambda) - E(\lambda)) \bar{P}_0|_{\text{Ran}(\bar{P}_0)}]^{-1} \bar{P}_0 V P_0 \psi_0.$$

*Proof.* The eigenvalue equation  $H(\lambda)\psi(\lambda) = E(\lambda)\psi(\lambda)$  can be split into the equivalent system of equations

$$P_0(\lambda V + E_0 - E(\lambda))P_0\psi(\lambda) + \lambda P_0 V \bar{P}_0 \psi(\lambda) = 0 \tag{3.13a}$$

$$\lambda \bar{P}_0 V P_0 \psi(\lambda) + \bar{P}_0 (H(\lambda) - E(\lambda)) \bar{P}_0 \psi(\lambda) = 0, \tag{3.13b}$$

by applying the projections  $P_0$  and  $\bar{P}_0$  respectively. From (3.13a) we learn that

$$\frac{E(\lambda) - E_0}{\lambda} \langle \psi_0, P_0 \psi(\lambda) \rangle - \langle \psi_0, V P_0 \psi(\lambda) \rangle = \langle V \psi_0, \bar{P}_0 \psi(\lambda) \rangle = o(1),$$

i.e.

$$\frac{E(\lambda) - E_0}{\lambda} \xrightarrow{\lambda \rightarrow 0} \langle \psi_0, V \psi_0 \rangle.$$

This shows the claim for  $n = 1$ . We show the lemma by induction. Suppose the claim holds for  $n$  and the assumptions of the lemma hold for  $n + 1$ . That is we can write

$$E(\lambda) = \sum_{k=0}^n \lambda^k E_k + \lambda^n R_n(\lambda)$$

with  $R_n(\lambda) = o(1)$  as  $\lambda \downarrow 0$ . In particular, this implies for the recursively defined functions

$$\begin{aligned} E^{[0]}(\lambda) &:= E(\lambda) \\ E^{[k]}(\lambda) &:= \frac{E^{[k-1]}(\lambda) - E_{k-1}}{\lambda} \end{aligned} \quad (3.14)$$

that

$$\lim_{\lambda \downarrow 0} E^{[k]}(\lambda) = E_k, \quad k = 0, \dots, n.$$

We write the part  $\bar{P}_0\psi(\lambda)$  as follows

$$\bar{P}_0\psi(\lambda) = \frac{\bar{P}_0}{H_0 - E_0}(H_0 - E_0)\bar{P}_0\psi(\lambda) = \frac{\bar{P}_0}{H_0 - E_0}[H(\lambda) - E(\lambda) + (E(\lambda) - E_0 - \lambda V)]\bar{P}_0\psi(\lambda).$$

Equation (3.13b) implies

$$\bar{P}_0\psi(\lambda) = \lambda \frac{\bar{P}_0}{H_0 - E_0} [-V P_0\psi(\lambda) + (E^{[1]}(\lambda) - V)\bar{P}_0\psi(\lambda)]. \quad (3.15)$$

Iterated insertion of (3.15) into itself, terminating the expansion after we have reached order  $\lambda^n$ , this leads to the following claim.

**Claim:** We have for  $k = 1, \dots, n$

$$P_0 V \bar{P}_0\psi(\lambda) = P_0 V \sum_{j=1}^k -\lambda^j K_{j-1} V P_0\psi(\lambda) + P_0 V \lambda^k R_k(\lambda) \bar{P}_0\psi(\lambda), \quad (3.16)$$

where  $R_k(\lambda)$  is defined by

$$R_k(\lambda) := \sum_{j=1}^k K_{j-1} (E^{[k+1-j]}(\lambda) - \delta_{jk} V). \quad (3.17)$$

(We note that expressions are well defined by the assumption  $\bar{P}_0 V \psi_0 \in \text{dom}(K_l)$ ). Let us now show the claim. Equation (3.16) for  $k = 1$  is just Equation (3.15). Assume that (3.16) is true for a specific  $k \leq n - 1$ . In this case, we insert first the Definition (3.17) and then Definition (3.14)

$$\begin{aligned} P_0 V R_k(\lambda) \bar{P}_0\psi(\lambda) &= P_0 V \sum_{j=1}^k K_{j-1} (E^{[k+1-j]}(\lambda) - \delta_{jk} V) \bar{P}_0\psi(\lambda) \\ &= P_0 V \sum_{j=1}^n (K_{j-1} (E_{k+1-j} - \delta_{jk} V) \bar{P}_0\psi(\lambda) + \lambda K_{j-1} E^{[k+2-j]}(\lambda) \bar{P}_0\psi(\lambda)). \end{aligned}$$

We now use (3.15) for the first summand and obtain

$$\begin{aligned} P_0 V R_k(\lambda) \bar{P}_0 \psi(\lambda) &= \lambda P_0 V \sum_{j=1}^k \left( K_{j-1} (E_{k+1-j} - \delta_{jk} V) K_0 (-V P_0 \psi(\lambda) + (E^{[1]}(\lambda) - V) \bar{P}_0 \psi(\lambda)) \right. \\ &\quad \left. + K_{j-1} E^{[k+2-j]}(\lambda) \bar{P}_0 \psi(\lambda) \right). \end{aligned}$$

Using (3.12) we find

$$\begin{aligned} P_0 V R_k(\lambda) \bar{P}_0 \psi(\lambda) &= \lambda P_0 V \left( K_k (-V P_0 \psi(\lambda) + (E^{[1]}(\lambda) - V) \bar{P}_0 \psi(\lambda)) \right. \\ &\quad \left. + \sum_{j=1}^k K_{j-1} E^{[k+2-j]}(\lambda) \bar{P}_0 \psi(\lambda) \right) \\ &= -\lambda P_0 V K_k V P_0 \psi(\lambda) + \lambda P_0 V \sum_{j=1}^{k+1} K_{j-1} (E^{[k+2-j]}(\lambda) - \delta_{j,k+1} V) \bar{P}_0 \psi(\lambda). \end{aligned}$$

By (3.17) this expression agrees with (3.16) with  $k$  replaced by  $k+1$ . This shows the claim.

Next we insert the claim for  $k = n$  into (3.13a) to conclude

$$\left( P_0 (E^{[1]}(\lambda) - V) P_0 + \sum_{j=1}^n \lambda^j P_0 V K_{j-1} V P_0 \right) P_0 \psi(\lambda) = \lambda^n P_0 V R_n(\lambda) \bar{P}_0 \psi(\lambda). \quad (3.18)$$

Taking the inner product with  $\psi_0$  and using the induction hypothesis (3.12), we obtain

$$E^{[1]}(\lambda) - E_1 - \sum_{j=1}^n \lambda^j E_{j+1} = \lambda^n \langle \psi_0, P_0 V R_n(\lambda) \bar{P}_0 \psi(\lambda) \rangle = \lambda^n \langle R_n(\lambda) V \psi_0, \bar{P}_0 \psi(\lambda) \rangle.$$

Dividing by  $\lambda^n$  we find using (3.14)

$$\lambda^{-(n+1)} \left( E(\lambda) - \sum_{j=0}^{n+1} \lambda^j E_j \right) = \langle R_n(\lambda) V \psi_0, \bar{P}_0 \psi(\lambda) \rangle = o(1).$$

This shows the claim of the lemma for  $n+1$ . □

*Remark 6.* Note that (3.18) implies

$$\left( E(\lambda) - \sum_{j=0}^n \lambda^j H_j \right) P_0 \psi(\lambda) = o(\lambda^n),$$

for  $H_1 := P_0 V P_0$  and  $H_n := -P_0 V K_{n-2} V P_0$ . This can be used for a degenerate perturbation theory, where each operator  $H_j$  has to be diagonalized and the coefficients  $E_j$  can be chosen out of the eigenvalues of  $H_j$ .

## 4 Ground State Energy

The main goal of this section is to show Theorem 1. To this end we first introduce notation in order to be able to write the expansion coefficients of the ground state energy as a sum of so called linked contractions involving renormalized propagators, which we call renormalized Feynman graphs. The renormalized propagators take into account the cancellations which results in an improved infrared behaviour. The basic idea behind this mechanism has been outlined in Remark 2 above. In the second part of this section we estimate the renormalized Feynman graphs and prove the finiteness of each expansion coefficient.

As a corollary of the proof of Theorem 1, we obtain a formula for the energies in terms of linked contractions and renormalized propagators. The formula will be given in Corollary 1.

For notational convenience we introduce the usual bosonic creation operators  $a^*(k)$  and annihilation operators  $a(k)$  satisfying canonical commutation relations

$$[a(k), a(k')] = 0, \quad [a^*(k), a^*(k')] = 0, \quad [a(k), a^*(k')] = \delta(k - k'),$$

for all  $k, k' \in \mathbb{R}^3$ . Using creation and annihilation operators we can write

$$a^*(G) = \int G(k) a^*(k) dk, \quad a(G) = \int G^*(k) a(k) dk.$$

Since we do not yet know the values of the energies  $E_n$  (indeed at this stage we do not even know their existence), we shall write in their place  $\mathcal{E}_n$ . At the end we will inductively construct the energies  $E_n$  as the value of a limit.

### 4.1 Graph functions and Substitutions

In this section we introduce notation which will later be needed.

**Definition 1.** Let  $G = (V, E)$  be a graph. Then a graph  $G_1 = (V_1, E_1)$  is called a **subgraph** of  $G$  and we write  $G_1 \subset G$ , if  $V_1 \subset V$  and  $E_1 \subset E$ . For a subset  $V_1 \subset V$  we define the **restricted graph**

$$G|_{V_1} := (V_1, \{e \in E : e \subset V_1\}).$$

We define the union of two graphs  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  by

$$G_1 \cup G_2 := (V_1 \cup V_2, E_1 \cup E_2).$$

For a subset  $X \subset \mathbb{Z}$  we associated the graph  $G_X = (X, E_X)$  with edges  $E_X$  consisting of nearest neighbor pairs of  $X$ , that is

$$E_X := \{\{x, r_X(x)\} : x \in X \setminus \max X\},$$

where  $r_X(x) := \min(X \setminus (-\infty, x])$  denotes the nearest neighbor vertex of  $x$  which lies to the right. We will also consider graphs with external lines, that is

$$\bar{G}_X := (X, \bar{E}_X)$$

with

$$\bar{E}_X := E_X \cup \{\{-\infty, \min X\}, \{\max X, \infty\}\}.$$

In this subsection let  $\mathcal{V}$  and  $\mathcal{R}$  denote two sets. Later we will refer to elements in  $\mathcal{V}$  as interactions and to elements in  $\mathcal{R}$  as resolvents or propagators.

**Definition 2.** Let  $X \subset \mathbb{Z}$ . For  $E = E_X$  and  $E = \bar{E}_X$  a function on  $(X, E)$  of the form

$$((V_x)_{x \in X}, (F_e)_{e \in E}),$$

where  $V_x \in \mathcal{V}$  for every  $x \in X$  and  $F_e \in \mathcal{R}$  for every  $e \in E$  is called a  $(\mathcal{V}, \mathcal{R})$ -valued **graph function on  $X$**  and a  $(\mathcal{V}, \mathcal{R})$ -valued **graph function on  $X$  with external lines**, respectively.

*Example 1.* We can write a graph function on  $\{1, 2, 3, 4, 5\}$  symbolically as

$$\begin{array}{ccccccccc} V_1 & F_{\{1,2\}} & V_2 & F_{\{2,3\}} & V_3 & F_{\{3,4\}} & V_4 & F_{\{4,5\}} & V_5 \\ \textcircled{1} & & \textcircled{2} & & \textcircled{3} & & \textcircled{4} & & \textcircled{5} \end{array}$$

Now we introduce a so called substitution operation, which substitutes a piece of a graph function by a simpler expression. This will later be used to express so called renormalized Feynmann graphs.

**Definition 3.** Let  $X \subset \mathbb{Z}$ , and let  $K \in \mathcal{R}$ . Let  $\pi = ((V_x)_{x \in X}, (F_e)_{e \in \bar{E}_X})$  be a  $(\mathcal{V}, \mathcal{R})$ -valued graph function with external lines on  $X$ . For  $I \subset X$  with  $G_I \subset G_X$  we define

$$\text{subst}_{I \rightarrow K}(\pi) := ((V_x)_{x \in X \setminus I}, (\tilde{F}_e)_{e \in \bar{E}_{X \setminus I}}), \quad (4.1)$$

where for  $e \in \bar{E}_{X \setminus I}$

$$\tilde{F}_e := \begin{cases} F_{\{\min e, \min I\}} K F_{\{\max e, \max I\}}, & e \notin \bar{E}_X, \\ F_e, & e \in \bar{E}_X \end{cases}$$

Note that (4.1) is again a  $(\mathcal{V}, \mathcal{R})$ -valued graph function on  $X \setminus I$  with external lines, that is, for a graph function with external lines we can substitute any subgraph and we obtain again a graph function with external lines. In Section 4.2 we show how Definition 3 can be naturally extend to graph functions without external lines.

*Example 2.* Let  $\pi$  denote the graph function of the previous example. Suppose  $I = \{2, 3\}$ . Then we write the graph function  $\text{subst}_{I \rightarrow K}(\pi)$  symbolically as

$$\begin{array}{ccccccc} V_1 & F_{\{1,2\}} & K & F_{\{3,4\}} & V_4 & F_{\{4,5\}} & V_5 \\ \textcircled{1} & & & & \textcircled{4} & & \textcircled{5} \end{array}.$$

The following lemma is a direct consequence of the definition.

**Lemma 5.** (*Commutativity*) Let  $X \subset \mathbb{Z}$  and let  $F_I, F_J \in \mathcal{R}$ . Let  $\pi$  be a  $(\mathcal{V}, \mathcal{R})$ -valued graph function on  $X$  with external lines. For any disjoint subsets  $I, J$  of  $X$  with  $G_I, G_J \subset G_X$ , we have

$$\text{subst}_{I \rightarrow F_I}(\pi) \text{subst}_{J \rightarrow F_J}(\pi) = \text{subst}_{J \rightarrow F_J}(\pi) \text{subst}_{I \rightarrow F_I}(\pi)$$

Lemma 5 justifies the use of the following notation. Let  $X \subset \mathbb{N}$  and let a set  $\mathcal{I}$  of mutually disjoint subsets of  $X$  be given such that  $G_I \subset G_X$  for all  $I \in \mathcal{I}$ , and let for each  $I \in \mathcal{I}$  an element  $F_I \in \mathcal{R}$  be given. Then we write for any  $(\mathcal{V}, \mathcal{R})$ -valued graph functions  $\pi$  on  $X$  with external lines

$$\text{subst}_{\substack{I \rightarrow F_I \\ I \in \mathcal{I}}}(\pi) := \prod_{I \in \mathcal{I}} \{\text{subst}_{I \rightarrow F_I}\}(\pi).$$

## 4.2 Operator Products

In this subsection we use the above notation to write the energy coefficients in terms of expectation values of operator products. To this end we introduce the set of interactions and the set of propagators suitable for the generalized spin boson model

$$\begin{aligned}\mathcal{V}_{\text{sb}} &= \{a^*(G) + a(G) : G \in \mathcal{L}(\mathbb{R}^d; \mathcal{L}(\mathcal{H}_{\text{at}}), (1 + |k|^{-2})dk)\} \\ \mathcal{R}_{\text{sb}} &= \{R : [0, \infty) \rightarrow \mathcal{L}(\mathcal{H}_{\text{at}}) \text{ piecewise continuous}\}.\end{aligned}$$

If we are given a  $(\mathcal{V}_{\text{sb}}, \mathcal{R}_{\text{sb}})$ -valued graph function on  $X$  with no external lines, we can naturally extend it to a graph function with external lines by assigning the identity operator in  $\mathcal{H}_{\text{at}}$  to each external line. With this extension we can naturally extend every definition for graph functions with external lines to such without external lines. In particular we can extend Definition 3 to graph functions without external lines.

For a finite set  $X \subset \mathbb{Z}$  and for  $\phi = ((V_x)_{x \in X}, (F_e)_{e \in \bar{E}_X})$  a  $(\mathcal{V}_{\text{sb}}, \mathcal{R}_{\text{sb}})$ -valued graph function on  $X$  with external lines, we define the formal operator product

$$\Pi(\phi) = F_{\{-\infty, \min X\}}(H_f) \prod_{x \in X \setminus \max X} \{V_x F_{\{x, r_X(x)\}}(H_f)\} V_{\max X} F_{\{\max X, \infty\}}(H_f).$$

Moreover, we define an energy shift

$$\mathcal{T}_r(\phi) := ((V_x)_{x \in X}, (F_e(\cdot + r))_{e \in \bar{E}_X}),$$

for  $r \geq 0$ . Let us now define a special graph function, which we will eventually use to write the expansion coefficients of the energy. For  $r \geq 0$  and  $\eta \geq 0$  define

$$R(r, \eta) := \frac{1 - P_{\text{at}} \otimes 1_{r=0}}{E_0 - H_{\text{at}} - r - \eta}. \quad (4.2)$$

We note that (4.2) is bounded if  $\eta > 0$  or  $r > 0$ . The parameter  $\eta$  serves as a regularization which we shall later remove. The parameter  $r$  will be needed to account for the additional terms arising from the pull-through formula.

Let us now define the graph functions which we will use for our model. We write

$$N_n := \mathbb{N} \cap [1, n],$$

and we define for  $r, \eta \geq 0$

$$\begin{aligned}\pi_n(\eta) &:= ((V)_{x \in N_n}, (R(\cdot, \eta))_{e \in E_{N_n}}) \\ \pi_n(r, \eta) &:= ((V)_{x \in N_n}, (R(\cdot + r, \eta))_{e \in E_{N_n}}) = \mathcal{T}_r \pi_n(\eta),\end{aligned}$$

where  $V \in \mathcal{V}_{\text{sb}}$  is the interaction of the spin boson model.

For a given sequence  $(\mathcal{E}_n)_{n \in \mathbb{N}}$  in  $\mathbb{C}$  we define the expression

$$T_n(r, \eta) := \sum_{k=1}^n \sum_{\substack{\{I_1, \dots, I_k\} \\ I_i \cap I_j = \emptyset, I_j \neq \emptyset \\ G_{I_i} \subsetneq G_{N_n}}} (1 \otimes P_\Omega) \Pi\left(\text{subst}_{\substack{I \rightarrow -\mathcal{E}_{|I|} \\ I \in \{I_1, \dots, I_k\}}}(\pi_n(r, \eta))\right) (1 \otimes P_\Omega), \quad (4.3)$$

where the second sum is over all sets with  $k$ -elements, with elements being nonzero disjoint subsets of  $N_n$  such that their associated graphs are subgraphs of  $G_{N_n}$  (this condition is imposed to ensure that  $I_k$  does not contain any holes). We shall adopt the convention that we view (4.3) as an operator restricted to the atomic Hilbert space. We also introduce the expressions

$$\widehat{T}_n(r, \eta) := T_n(r, \eta) - \mathcal{E}_n,$$

which we will refer to as renormalized propagators. Henceforth we shall write  $P_\Omega$  instead of  $1 \otimes P_\Omega$ . In the following lemma we relate the energy formula (2.8) in Theorem 1 to the expressions defined in (4.3).

**Lemma 6.** *Suppose  $\eta > 0$ . Then for any sequence  $(\mathcal{E}_n)_{n \in \mathbb{N}}$  of real numbers we have*

$$\begin{aligned} & \langle \varphi_{\text{at}}, T_n(0, \eta) \varphi_{\text{at}} \rangle \\ &= \sum_{k=2}^n \sum_{\substack{j_1 + \dots + j_k = n \\ j_s \geq 1}} \langle \psi_0, (\delta_{1j_1} V - \mathcal{E}_{j_1}) \prod_{s=2}^k \{ (E_0 - \eta - H_0)^{-1} \bar{P}_0 (\delta_{1j_s} V - \mathcal{E}_{j_s}) \} \psi_0 \rangle. \end{aligned}$$

*Proof.* To see this, we identify each summand in the sum. Consider the summand in (4.3) indexed by  $\mathcal{I} = \{I_1, \dots, I_l\}$ . We complement this set by sets consisting of elements of  $N_n$  which are not contained in any of the sets in  $\mathcal{I}$ . To this end we define

$$\mathcal{J} := \{\{s\} : s \in N_n \text{ and } s \notin I, \forall I \in \mathcal{I}\}.$$

Now we order the elements of  $\mathcal{S} := \mathcal{I} \cup \mathcal{J}$  in increasing order in the sense that for all  $s_1, s_2 \in \mathcal{S}$  we set

$$s_1 < s_2 :\Leftrightarrow \text{every element of } s_2 \text{ is an upper bound of } s_1.$$

This defines a bijection  $\varphi : N_{|\mathcal{S}|} \rightarrow \mathcal{S}$  preserving the order. By construction we see that the summand in (4.3) indexed by  $\{I_1, \dots, I_l\}$  is equal to the summand in (2.8), which we obtain by choosing  $k = |\mathcal{S}|$ , the indices  $j_s = |\varphi(s)|$  for  $s = 1, \dots, k$ , by choosing  $-\mathcal{E}_1$  in case  $j_s = 1$  and  $\varphi(s) \in \mathcal{I}$ , and by choosing  $V$  if  $j_s = 1$  and  $\varphi(s) \notin \mathcal{I}$ .  $\square$

Finally we give an alternative formulation for (4.3), to shorten the notation in forthcoming proofs. For  $S \subset \mathbb{Z}$  we say that a set of the form  $\{s \in S : a \leq s \leq b\}$  for some  $a, b \in S$  is an interval of  $S$ . For  $M \subset \mathbb{Z}$  we define the set  $\mathcal{Q}(M)$  consisting of all collections of disjoint nonempty intervals of  $M$ , i.e.,

$$\begin{aligned} \mathcal{Q}(M) &:= \{\mathcal{I} \subset \mathcal{P}(M) : \forall I, J \in \mathcal{I} \text{ we have } I \cap J = \emptyset, \\ &\quad \forall I \in \mathcal{I} \text{ the set } I \text{ is a nonempty interval of } M\}. \end{aligned}$$

Then we can rewrite (4.3) as

$$T_n(r, \eta) = \sum_{\substack{\mathcal{I} \in \mathcal{Q}(N_n) \\ N_n \notin \mathcal{I}}} P_\Omega \Pi(\text{subst}_{I \rightarrow -\mathcal{E}_{|I|}}(\pi_n(r, \eta))) P_\Omega, \quad (4.4)$$

and for the renormalized expression

$$\widehat{T}_n(r, \eta) = \sum_{\mathcal{I} \in \mathcal{Q}(N_n)} P_\Omega \Pi(\text{subst}_{I \rightarrow -\mathcal{E}_{|I|}}(\pi_n(r, \eta))) P_\Omega. \quad (4.5)$$



### 4.3 Wicks Theorem and Contractions

Now we use a generalized version of Wicks theorem to write (4.3) as a sum of so called contractions. To this end we introduce the following notation.

**Definition 4.** Let  $X$  be a finite set. A **pair** of  $X$  is a subset of  $X$  containing two elements. A **pair partition**  $P$  of  $X$  is a partition of  $X$  consisting of pairs of  $X$ , i.e.,  $|X|$  is even we have

$$P = \{p_1, p_2, \dots, p_{\frac{|X|}{2}}\}$$

where  $p_j$  is a pair of  $X$ ,  $p_i \cap p_j = \emptyset$  if  $i \neq j$ , and  $\bigcup_{p \in P} p = X$ . A **pairing** of  $X$  is a pair partition of a subset of  $X$ .

**Definition 5.** Let  $X \subset \mathbb{Z}$  be finite, let  $P$  be a pairing of  $X$ , and let  $\phi = ((a(G_x) + a^*(G_x))_{x \in X}, (F_e)_{e \in \bar{E}_X})$  be a  $(\mathcal{V}_{\text{sb}}, \mathcal{R}_{\text{sb}})$ -valued graph function with external lines on  $X$ . Then the **contraction** of  $\phi$  with respect to  $P$  is defined by

$$\begin{aligned} \mathcal{C}_P(\phi)(r) &:= \prod_{j \in X} \left\{ \int dk_j \right\} \delta_P(k) F_{\{-\infty, \min X\}}(r) \\ &\times \prod_{j \in X \setminus \{\max X\}} \left\{ G_{j,P}^\#(k_j) F_{\{j, r_X(j)\}}(|K_{\{j, r_X(j)\}}|_P + r) \right\} I_{\max X, P}(k_{\max X}) F_{\{\max X, \infty\}}(r), \quad r \geq 0, \end{aligned}$$

where

$$\begin{aligned} G_{j,P}^\# &:= \begin{cases} G_j^* & , \exists p \in P, j = \min p \\ G_j & , \exists p \in P, j = \max p \end{cases} \\ \delta_P(k) &:= \prod_{p \in P} \delta(k_{\min p} - k_{\max p}) \\ |K_e|_P &:= \sum_{\substack{p \in P \\ \max e \leq \max p \\ \min p \leq \min e}} |k_{\max p}| \end{aligned}$$

We define

$$\mathcal{C}_P^0(\phi) := \mathcal{C}_P(\phi)(0).$$

We shall adopt the following conventions. We write

$$\mathcal{C}(\phi) := \sum_P \mathcal{C}_P(\phi),$$

where the sum is over all pair partitions of  $X$ . If  $P$  is not a pair partition of  $X$  we set for notational compactness

$$\mathcal{C}_P(\phi) = 0.$$

We adopt analogous conventions for  $\mathcal{C}^0$ .

**Lemma 7.** Suppose the situation is as in Definition 5, and that for all  $e \in \bar{E}_X$  the functions  $F_e(r)$  are uniformly bounded in  $r \geq 0$ . Then  $\mathcal{C}(\phi) \in \mathcal{R}_{\text{sb}}$ .

*Proof.* This follows from the dominated convergence theorem.  $\square$

*Remark 7.* Let the situation be as in Definition 5. Then we have

$$\mathcal{C}_P(\phi)(r) = \mathcal{C}_P^0(\mathcal{T}_r\phi).$$

We illustrate Definition 5 in the following example.

*Example 3.* Consider the set  $N_4$  and we consider the pair partition  $P$  which is indicated by the lines below.

$$P : \quad \begin{array}{c} \circ \quad \circ \quad \circ \quad \circ \\ \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \end{array}$$

$$\begin{aligned} \mathcal{C}_P(\phi)(r) = \int dk_1 \cdots dk_4 & \delta(k_1 - k_3) \delta(k_2 - k_4) F_{\{-\infty, 1\}}(r) G^*(k_1) F_{\{1, 2\}}(|k_3| + r) G^*(k_2) \\ & F_{\{2, 3\}}(|k_3| + |k_4| + r) G(k_3) F_{\{3, 4\}}(|k_4| + r) G(k_4) F_{\{4, \infty\}}(r). \end{aligned}$$

**Lemma 8** (Generalized Wick Theorem). *Let  $X \subset \mathbb{N}$  be finite and let  $\phi$  be a  $(\mathcal{V}_{\text{sb}}, \mathcal{R}_{\text{sb}})$ -valued graph function on  $G_X$  or  $\bar{G}_X$ . Then*

$$P_\Omega \Pi(\phi) P_\Omega = \mathcal{C}^0(\phi)$$

The proof follows from the usual Wick theorem, leaving the operator valued functions  $G$  at their position and using the so called pull through formula. The pull through formula gives the commutation relation between the free field energy and the creation or annihilation operators. For a detailed proof we refer the reader to [5]. The following lemma is an immediate consequence of (4.4) resp. (4.5) and the generalized Wick theorem (Lemma 8).

**Lemma 9.** *We have*

$$T_n(r, \eta) = \sum_{\substack{\mathcal{I} \in \mathcal{Q}(N_n) \\ N_n \notin \mathcal{I}}} \mathcal{C}^0(\text{subst}_{I \rightarrow -\mathcal{E}_{|I|}}(\pi_n(r, \eta))) \quad (4.6)$$

$$\hat{T}_n(r, \eta) = \sum_{\mathcal{I} \in \mathcal{Q}(N_n)} \mathcal{C}^0(\text{subst}_{I \rightarrow -\mathcal{E}_{|I|}}(\pi_n(r, \eta))). \quad (4.7)$$

We shall need the following lemma which collects two algebraic relations of the contraction operation.

**Lemma 10.** *Let  $X \subset \mathbb{Z}$  be finite, let  $P$  be a pairing of  $X$ .*

- (a) *For  $j = 1, 2$  let  $\phi_j = ((V_x)_{x \in X}, (F_{j,e})_{e \in \bar{E}_X})$  be  $(\mathcal{V}_{\text{sb}}, \mathcal{R}_{\text{sb}})$ -valued graph functions with external lines on  $X$  and suppose for  $e \in \bar{E}_X$  we are given numbers  $t_{e,j} \in \mathbb{C}$ . Then the following multilinearity relation holds*

$$\mathcal{C}_P((V_x)_{x \in X}, (\sum_{j_e=1,2} t_{e,j_e} F_{j_e,e})_{e \in \bar{E}_X}) = \prod_{e \in \bar{E}_X} \left\{ \sum_{j_e=1,2} t_{e,j_e} \right\} \mathcal{C}_P((V_x)_{x \in X}, (F_{j_e,e})_{e \in \bar{E}_X}).$$

(b) Suppose we are given disjoint sets  $X_l, X_r \subset X$  such that their union equals  $X$  and

$$\max X_l < \min X_r.$$

Furthermore, assume that  $P = P_l \cup P_r$  where  $P_l$  is a pair partition of  $X_l$  and  $P_r$  is a pair partition of  $X_r$ . Then for any  $(\mathcal{V}_{\text{sb}}, \mathcal{R}_{\text{sb}})$ -valued graph function,  $\phi = ((V_x)_{x \in X}, (F_e)_{e \in E_X})$ , we have the

$$\mathcal{C}_P(\phi) = \mathcal{C}_{P_l}(\phi_l) F_{\{\max X_l, \min X_r\}} \mathcal{C}_{P_r}(\phi_r),$$

where we defined

$$\phi_l := ((V_x)_{x \in X_l}, (F_e)_{e \in E_{X_l}}), \quad \phi_r := ((V_x)_{x \in X_r}, (F_e)_{e \in E_{X_r}}).$$

(c) Suppose we are given disjoint sets  $X_l, X_m, X_r \subset X$  such that their union equals  $X$  and

$$\max X_l < \min X_m < \max X_m < \min X_r.$$

Furthermore, assume that  $P = P_m \cup P_b$  where  $P_m$  is a pair partition of  $X_m$  and  $P_b$  is a pair partition of  $X_l \cup X_r$ . Then for,  $\phi = ((V_x)_{x \in X}, (F_e)_{e \in E_X})$ , any  $(\mathcal{V}_{\text{sb}}, \mathcal{R}_{\text{sb}})$ -valued graph functions with external lines on  $X$  we have the following substitution relation

$$\mathcal{C}_P(\phi) = \mathcal{C}_{P_b}(\text{subst}_{X_m \rightarrow \tilde{\phi}}(\phi)),$$

where

$$\tilde{\phi} := \mathcal{C}_{P_m}((V_x)_{x \in X_m}, (F_e)_{e \in E_{X_m}}).$$

*Proof.* (a). This follows from the bilinearity of the operator product and the linearity of the integral. Statements (b) and (c) follow from Fubini's Theorem.  $\square$

## 4.4 Feynman Graphs, Renormalization

In this subsection we want to evaluate the sum over all contractions in (4.6). To this end we will show that we can write (4.6) as a sum of so called linked contractions over so called renormalized propagators. We shall use the notation, that for a set  $\mathcal{A}$  of sets we write  $\bigcup \mathcal{A} := \bigcup_{A \in \mathcal{A}} A$ .

**Definition 6.** Let  $X \subset \mathbb{Z}$ . We call two distinct elements  $p_1$  and  $p_2$  of a pairing of  $X$  **linked** if one element of  $p_1$  lies between the elements of  $p_2$  and one of the elements of  $p_2$  lies between the elements of  $p_1$ , i.e.,

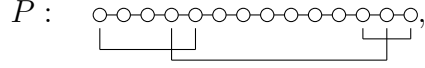
$$p_1 \cap [\min p_2, \max p_2] \neq \emptyset \quad \text{and} \quad p_2 \cap [\min p_1, \max p_1] \neq \emptyset.$$

For a pairing  $P$  of  $X$ , we call the mapping

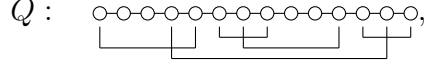
$$\gamma : \{0, \dots, l\} \rightarrow P,$$

with  $l \in \mathbb{N}$ , a **linked path** in  $P$  from  $\gamma(0)$  to  $\gamma(l)$  of **length**  $l$  if  $\gamma(i)$  and  $\gamma(i+1)$  are linked for all  $i = 0, \dots, l-1$ . A pairing  $P$  of  $X$  is called **linked** if for any two elements  $p_1, p_2 \in P$  there exists a linked path in  $P$  from  $p_1$  to  $p_2$ . The property that there exists a linked path between two pairings is an equivalence relation on  $P$ , and we call the equivalence classes **linked components** of  $P$ . We say that  $P$  **links two elements**  $x, y$  of  $X$  if  $P$  has a linked component  $P_0$  such that  $x, y \in \bigcup P_0$ .

*Example 4.* Consider  $N_{14}$ . Then the pairing  $P = \{\{1, 5\}, \{4, 13\}, \{12, 14\}\}$  is linked



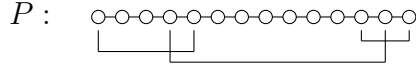
and the pairing  $Q = \{\{1, 5\}, \{4, 13\}, \{6, 8\}, \{7, 11\}, \{12, 14\}\}$ ,



can be written as the union of the linked components  $P$  and  $\{\{6, 8\}, \{7, 11\}\}$ .

Next we consider the set of non-paired elements. Specifically, for any pairing  $P$  of  $N_n$  we define  $\mathcal{I}_P$  to be coarsest partition of the the set of all partitions of  $N_n \setminus \bigcup P$  into intervals of  $N_n$ . This definition is illustrated in the next example.

*Example 5.* Consider  $N_{14}$  and  $P = \{\{1, 5\}, \{4, 13\}, \{12, 14\}\}$ .



Then  $\mathcal{I}_P = \{\{2, 3\}, \{6, 7, 8, 9, 10, 11\}\}$ .

*Remark 8.* Let us characterize  $\mathcal{I}_P$  in different terms. The set  $\mathcal{I}_P$  is the unique partition of  $N_n \setminus \bigcup P$  such that  $\bigcup_{I \in \mathcal{I}_P} G_I = G_{N_n}|_{N_n \setminus \bigcup P}$ .

We define

$$\begin{aligned} C_n(r, \eta) &:= \sum_{\substack{P \text{ pairing of } N_n \\ \{1, n\} \subset \bigcup P \\ P \text{ linked}}} \mathcal{C}_P(\text{subst}_{I \rightarrow \widehat{T}_I(\cdot, \eta)}(\pi_n(\eta))(r) \\ &= \sum_{\substack{P \text{ pairing of } N_n \\ \{1, n\} \subset \bigcup P \\ P \text{ linked}}} \mathcal{C}_P^0(\text{subst}_{I \rightarrow \widehat{T}_I(\cdot + r, \eta)}(\pi_n(r, \eta)), \end{aligned} \quad (4.8)$$

where we refer to the summand on the right hand side as a linked Feynman graph with renormalized propagators. We define  $\widehat{C}_n(r, \eta) := C_n(r, \eta) - \mathcal{E}_n$ .

**Proposition 1.** *We have*

$$T_n(r, \eta) = C_n(r, \eta) + \sum_{k=2}^n \sum_{\substack{j_1 + \dots + j_k = n \\ j_i \geq 1}} \left[ \prod_{i=1}^{k-1} (\widehat{C}_{j_i}(r, \eta) R(r, \eta)) \right] \widehat{C}_{j_k}(r, \eta).$$

For the proof we will introduce the notion of a connected pairings, which is similar to the notation of linked pairings, but not the same.

**Definition 7.** Let  $X \subset \mathbb{Z}$ . For a pairing  $P$  of  $X$  we call the set

$$S(P) = \bigcup_{p \in P} [p_1, p_2]$$

the span of  $P$ . We call two distinct elements  $p_1$  and  $p_2$  of a pairing of  $X$  **connected** if one element of  $p_1$  lies between the elements of  $p_2$  or one of the elements of  $p_2$  lies between the elements of  $p_1$ , i.e.,

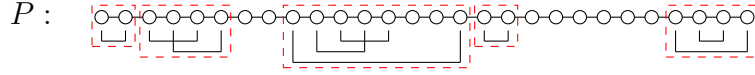
$$[\min p_1, \max p_1] \cap [\min p_2, \max p_2] \neq \emptyset.$$

For a pairing  $P$  of  $X$ , we call the mapping

$$\gamma : \{0, \dots, l\} \rightarrow P,$$

with  $l \in \mathbb{N}$ , a **connected path** in  $P$  from  $\gamma(0)$  to  $\gamma(l)$  if  $\gamma(i)$  and  $\gamma(i+1)$  are connected for all  $i = 0, \dots, l-1$ . A pairing  $P$  of  $X$  is called **connected** if for any two elements  $p_1, p_2 \in P$  there exists a connected path in  $P$  from  $p_1$  to  $p_2$ . The property that there exists a connected path between two pairings is an equivalence relation on  $P$ , and we call the equivalence classes **connected components** of  $P$ .

*Example 6.* Let  $P$  be a pairing, whose pairs are indicated by the black lines.



The connected components are indicated by the red boxes, that is a connected component consist of all the pairs in a single red box.

We note that linked implies connected but not the other way around.

*Proof of Proposition 1.* By Lemma 9 we can write

$$\begin{aligned} T_n(r, \eta) &= \sum_{P \text{ pairing of } N_n} \sum_{\substack{\mathcal{I} \in \mathcal{Q}(N_n) \\ N_n \notin \mathcal{I}}} \mathcal{C}_P^0(\text{subst}_{I \rightarrow -\mathcal{E}_{|I|}}(\pi_n(r, \eta))) \\ &= T_n^{(C)}(r, \eta) + T_n^{(D)}(r, \eta), \end{aligned}$$

where we divided the sum over the partitions into partitions which connect the smallest and the largest vertex

$$T_n^{(C)}(r, \eta) := \sum_{\substack{P \text{ pairing of } N_n \\ P \text{ links } 1, n}} \sum_{\substack{\mathcal{I} \in \mathcal{Q}(N_n) \\ N_n \notin \mathcal{I}}} \mathcal{C}_P^0(\text{subst}_{I \rightarrow -\mathcal{E}_{|I|}}(\pi_n(r, \eta)))$$

(observe that in the above formula we can drop the condition  $N_n \notin \mathcal{I}$  because of  $\{1, n\} \subset \bigcup P$ ) and the remaining partitions

$$T_n^{(D)}(r, \eta) := \sum_{\substack{P \text{ pairing of } N_n \\ P \text{ does not link } 1, n}} \sum_{\substack{\mathcal{I} \in \mathcal{Q}(N_n) \\ N_n \notin \mathcal{I}}} \mathcal{C}_P^0(\text{subst}_{I \rightarrow -\mathcal{E}_{|I|}}(\pi_n(r, \eta))). \quad (4.9)$$

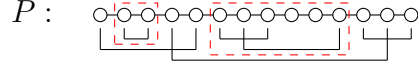
To simplify the connected part  $T_n^{(C)}(r, \eta)$  we proceed as follows. We decompose the pairing  $P$  of  $N_n$  which links 1 and  $n$ , into linked components. We denote the linked component which contains  $\{1, n\}$  by  $P_e$ . Each of the remaining linked components must be a pairing of  $I$  for some  $I \in \mathcal{I}_{P_e}$ , since otherwise the pairing would link two elements of  $N_n$ , for which there

would lie an element of  $\bigcup P_e$  between them, a contradiction. Thus we can write the pairing  $P$  of  $N_n$  which links 1 and  $n$  in a unique way as

$$P = P_e \cup \bigcup_{I \in \mathcal{I}_{P_e}} P_I, \quad (4.10)$$

where  $P_I$  is a pairing of  $I$ . This decomposition is illustrated in the following example.

*Example 7.* Consider  $N_{14}$  and let  $P$  be the set of the pairs indicated by black lines.



Then  $P_e$  is the set of all pairs indicated by the dashed lines which are outside of the dashed boxes. Moreover,  $\mathcal{I}_{P_e} = \{I_1, I_2\}$  with  $I_1 := \{2, 3\}$ , that is the set of points in first dashed box, and  $I_2 := \{6, 7, 8, 9, 10, 11\}$ , that is the set of points in the second dashed box. Furthermore  $P_{I_1}$  is the set of all pairs indicated by the lines in the first dashed box, and similarly for  $P_{I_2}$ .

Now using the decomposition (4.10) we obtain the first identity of the following equations

$$\begin{aligned} T_n^{(C)}(r, \eta) &= \sum_{\substack{P_e \text{ pairing of } N_n \\ \{1, n\} \subset \bigcup P_e \\ P_e \text{ linked}}} \prod_{I \in \mathcal{I}_{P_e}} \left\{ \sum_{P_I \text{ pairing of } I} \right\} \sum_{\mathcal{I} \in \mathcal{Q}(N_n)} \mathcal{C}_{(P_e \cup \bigcup_{I \in \mathcal{I}_{P_e}} P_I)}^0 \left( \text{subst}_{\substack{J \rightarrow -\mathcal{E}_{|J|} \\ J \in \mathcal{I}}} (\pi_n(r, \eta)) \right) \\ &= \sum_{\substack{P_e \text{ pairing of } N_n \\ \{1, n\} \subset \bigcup P_e \\ P_e \text{ linked}}} \prod_{I \in \mathcal{I}_{P_e}} \left\{ \sum_{\mathcal{I}_I \in \mathcal{Q}(I)} \sum_{\substack{P_I \text{ pairing of } I \\ I \setminus \bigcup \mathcal{I}_I}} \right\} \mathcal{C}_{(P_e \cup \bigcup_{I \in \mathcal{I}_{P_e}} P_I)}^0 \left( \text{subst}_{\substack{J \rightarrow -\mathcal{E}_{|J|} \\ J \in \bigcup_{I \in \mathcal{I}_{P_e}} \mathcal{I}_I}} (\pi_n(r, \eta)) \right) \quad (4.11) \\ &= \sum_{\substack{P_e \text{ pairing of } N_n \\ \{1, n\} \subset \bigcup P_e \\ P_e \text{ linked}}} \mathcal{C}_{P_e}^0 \left( \text{subst}_{I \rightarrow \hat{T}_{|I|}(\cdot + r, \eta)} (\pi_n(r, \eta)) \right) \quad (4.12) \\ &= C_n(r, \eta), \end{aligned}$$

where in (4.11) we interchanged on each of the intervals  $I \in \mathcal{I}_{P_e}$  the summation on the one hand over energy subtractions and on the other hand over pairings of  $I$ , and where in (4.12) we used linearity of the contraction operator  $\mathcal{C}_P$  and the multilinearity property of the product of graph functions, see Lemma 10.

To simplify  $T_n^{(D)}(r, \eta)$  we rearrange the summation by decomposing it into disconnected parts. To this end we define a bijection between two index sets. The original index set is

$$\mathcal{S}_1 := \{(P, \mathcal{I}) : \mathcal{I} \subset \mathcal{Q}(N_n), P \text{ is a pair partition of } N_n \setminus \bigcup \mathcal{I}\}.$$

Now suppose  $(P, \mathcal{I}) \in \mathcal{S}_1$  is given. The following construction is illustrated in the example below. First we consider the connected components of  $P$ , and we define the sets

$$\begin{aligned} \mathcal{K}_0 &:= \{S(Q) \cap \mathbb{N} : Q \text{ connected component of } P\} \\ \mathcal{I}_0 &:= \{K \in \mathcal{I} : K \subset N_n \setminus S(P)\} \\ \mathcal{K} &:= \mathcal{K}_0 \cup \mathcal{I}_0. \end{aligned}$$

Clearly,  $\mathcal{K}$  is a partition of  $N_n$  into intervals of  $N_n$ . Next we want to decompose  $(P, \mathcal{I})$  with respect to the partition  $\mathcal{K}$ . For this we define for each  $K \in \mathcal{K}$  the set

$$\mathcal{I}_K := \{I \subset K : I \in \mathcal{I}\},$$

moreover, we define for  $K \in \mathcal{K}_0$  the pairing

$$P_K := Q,$$

where  $Q$  is the unique connected component of  $P$  such that  $K = S(Q) \cap \mathbb{N}$ , and for  $K \in \mathcal{I}_0$  we define  $P_K = \emptyset$ . It is straight forward to verify that this construction yields a well defined map  $\psi$  from the set  $\mathcal{S}_1$  to the index set

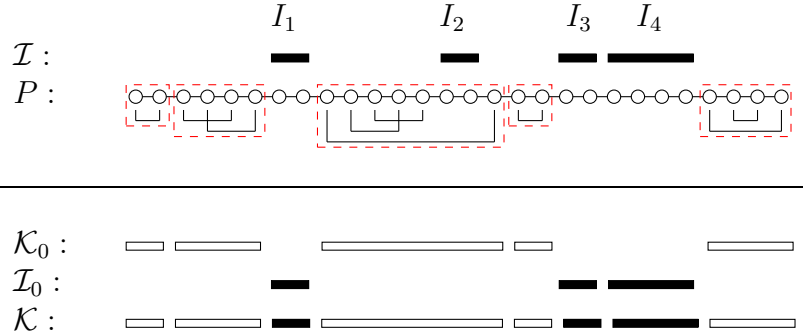
$$\begin{aligned} \mathcal{S}_2 := \{ & (\mathcal{K}, (P_K)_{K \in \mathcal{K}}, (\mathcal{I}_K)_{K \in \mathcal{K}}) : \mathcal{K} \in \mathcal{Q}(N_n), \bigcup \mathcal{K} = N_n, \\ & \mathcal{I}_K \in \mathcal{Q}(K), P_K \text{ is a pair partition of } K \setminus \bigcup \mathcal{I}_K, \\ & (S(P_K) = [\min K, \max K] \text{ or } (P_K = \emptyset \text{ and } \mathcal{I}_K = K))\}. \end{aligned}$$

In fact  $\psi$  is a bijection with inverse

$$\psi^{-1}(\mathcal{K}, (P_K)_{K \in \mathcal{K}}, (\mathcal{I}_K)_{K \in \mathcal{K}}) = (\bigcup_{K \in \mathcal{K}} P_K, \bigcup_{K \in \mathcal{K}} \mathcal{I}_K),$$

as one readily verifies.

*Example 8.* We consider a pairing  $P$  of  $N_{28}$ , whose pairs are indicated by the black lines below. The set  $\mathcal{I} = \{I_1, I_2, I_3, I_4\}$  with  $I_1 := \{7, 8\}$ ,  $I_2 := \{14, 15\}$ ,  $I_3 := \{19, 20\}$ ,  $I_4 := \{21, 22, 23, 24\}$  is indicated below as well. Likewise the sets  $\mathcal{K}_0$ ,  $\mathcal{I}_0$ , and  $\mathcal{K}$  are indicated.



Furthermore, one sees that

$$\psi(\{(P, \mathcal{I}) \in \mathcal{S}_1 : S(P) \neq [1, n], N_n \notin \mathcal{I}\}) = \{(\mathcal{K}, (P_K)_{K \in \mathcal{K}}, (\mathcal{I}_K)_{K \in \mathcal{K}}) \in \mathcal{S}_2 : |\mathcal{K}| \geq 2\}.$$

Thus the bijection allows us to rearrange the sum in (4.9) and we obtain the first identity of

the following equations

$$\begin{aligned}
T_n^{(D)}(r, \eta) &= \sum_{\substack{\mathcal{K} \in \mathcal{Q}(N_n) \\ \bigcup \mathcal{K} = N_n \\ |\mathcal{K}| \geq 2}} \prod_{K \in \mathcal{K}} \left\{ \sum_{\substack{P_K \text{ pairing of } K \\ S(P_K) = [\min K, \max K]}} \sum_{\mathcal{I}_K \in \mathcal{Q}(K)} + 1_{(P_K = \emptyset, \mathcal{I}_K = \{K\})} \right\} \mathcal{C}_{(\bigcup_{K \in \mathcal{K}} P_K)}^0 \left( \text{subst}_{\substack{I \rightarrow -\mathcal{E}_{|I|} \\ I \in \bigcup_{K \in \mathcal{K}} \mathcal{I}_K}} (\pi_n(r, \eta)) \right) \\
&= \sum_{\substack{\mathcal{K} \in \mathcal{Q}(N_n) \\ \bigcup \mathcal{K} = N_n \\ |\mathcal{K}| \geq 2}} \prod_{K \in \mathcal{K}} \left\{ \sum_{\substack{P_K \text{ pairing of } K \\ S(P_K) = [\min K, \max K]}} \sum_{\mathcal{I}_K \in \mathcal{Q}(K)} + 1_{(P_K = \emptyset, \mathcal{I}_K = \{K\})} \right\} \\
&\quad \times \prod_{K \in \mathcal{K} \setminus \max \mathcal{K}} \left\{ \mathcal{C}_{P_K}^0 \left( \text{subst}_{\substack{I \rightarrow -\mathcal{E}_{|I|} \\ I \in \mathcal{I}_K}} (\pi_{|K|}(r, \eta)) \right) R(r, \eta) \right\} \mathcal{C}_{P_{\max \mathcal{K}}}^0 \left( \text{subst}_{\substack{I \rightarrow -\mathcal{E}_{|I|} \\ I \in \mathcal{I}_{\max \mathcal{K}}}} (\pi_{|\max \mathcal{K}|}(r, \eta)) \right) \quad (4.13)
\end{aligned}$$

$$= \sum_{k=2}^n \sum_{\substack{j_1 + \dots + j_k = n \\ j_i \geq 1}} \left[ \prod_{i=1}^{k-1} (\widehat{C}_{j_i}(r, \eta) R(r, \eta)) \right] \widehat{C}_{j_k}(r, \eta), \quad (4.14)$$

where in (4.13) we made use of the fact that the contraction factors for disconnected parts and the product over  $\mathcal{K}$  is taken with respect to the following ordering. An element  $\{I_1, \dots, I_l\}$  of  $\mathcal{Q}(M)$  has an ordering given by

$$I_i \leq I_j \quad :\Leftrightarrow \quad a \leq b, \forall a \in I_i, \forall b \in I_j, \quad (4.15)$$

which is total and well ordered. Finally we note that Eq. (4.14) follows from multilinearity and the definitions.  $\square$

In order to estimate the Feynmann graphs, we will decompose the resolvent in Proposition 1. We write

$$R(r, \eta) = R^\perp(r, \eta) + R^\parallel(r, \eta), \quad (4.16)$$

where we have defined

$$R^\perp(r, \eta) = (1 - P_{\text{at}})R(r, \eta), \quad R^\parallel(r, \eta) = P_{\text{at}}R(r, \eta).$$

We define

$$G_n(r, \eta) := C_n(r, \eta) + \sum_{k=2}^n \sum_{\substack{j_1 + \dots + j_k = n \\ j_i \geq 1}} \left[ \prod_{i=1}^{k-1} (\widehat{C}_{j_i}(r, \eta) R^\perp(r, \eta)) \right] \widehat{C}_{j_k}(r, \eta) \quad (4.17)$$

and

$$\widehat{G}_n(r, \eta) := G_n(r, \eta) - \mathcal{E}_n.$$

The next theorem is purely algebraic. It will later be used to estimate the energy coefficients.



**Theorem 3.** *We have*

$$T_n(r, \eta) = G_n(r, \eta) + \sum_{k=2}^n \sum_{\substack{j_1+\dots+j_k=n \\ j_i \geq 1}} \left[ \prod_{i=1}^{k-1} (\widehat{G}_{j_i}(r, \eta) R^\parallel(r, \eta)) \right] \widehat{G}_{j_k}(r, \eta) \quad (4.18)$$

and

$$\widehat{T}_n(r, \eta) = \widehat{G}_n(r, \eta) + \sum_{k=2}^n \sum_{\substack{j_1+\dots+j_k=n \\ j_i \geq 1}} \left[ \prod_{i=1}^{k-1} (\widehat{G}_{j_i}(r, \eta) R^\parallel(r, \eta)) \right] \widehat{G}_{j_k}(r, \eta). \quad (4.19)$$

*Proof.* In view of the previous proposition it remains to decompose the resolvent between disconnected parts into orthogonal and parallel part. To this end we multiply out the expressions and collect the terms according to the number,  $s - 1$ , of resolvents with  $R^\parallel$ . Thus by straight forward algebraic calculation we obtain

$$\begin{aligned} & \sum_{k=2}^n \sum_{\substack{j_1+\dots+j_k=n \\ j_i \geq 1}} \left[ \prod_{i=1}^{k-1} (\widehat{C}_{j_i}(r, \eta) R(r, \eta)) \right] \widehat{C}_{j_k}(r, \eta) \\ &= \sum_{k=2}^n \sum_{\sigma_1, \dots, \sigma_{k-1} \in \{\perp, \parallel\}} \sum_{\substack{j_1+\dots+j_k=n \\ j_i \geq 1}} \left[ \prod_{i=1}^{k-1} (\widehat{C}_{j_i}(r, \eta) R^{\sigma_i}(r, \eta)) \right] \widehat{C}_{j_k}(r, \eta) \\ &= \sum_{s=1}^n \sum_{\substack{n_1+\dots+n_s=n \\ n_i \geq 1}} \sum_{\substack{k_1, \dots, k_s \in \mathbb{N}_n \\ k_1+\dots+k_s \geq 2}} \sum_{\substack{j_{1,1}+\dots+j_{1,k_1}=n_1 \\ j_{1,i} \geq 1}} \cdots \sum_{\substack{j_{s,1}+\dots+j_{s,k_s}=n_s \\ j_{s,i} \geq 1}} \\ & \quad \left[ \prod_{i_1=1}^{k_1-1} (\widehat{C}_{j_{1,i_1}}(r, \eta) R^\perp(r, \eta)) \right] \widehat{C}_{j_{1,k_1}}(r, \eta) R^\parallel(r, \eta) \cdots \left[ \prod_{i_s=1}^{k_s-1} (\widehat{C}_{j_{s,i_s}}(r, \eta) R^\perp(r, \eta)) \right] \widehat{C}_{j_{s,k_s}}(r, \eta) \\ &= \sum_{k=2}^n \sum_{\substack{j_1+\dots+j_k=n \\ j_i \geq 1}} \left[ \prod_{i=1}^{k-1} (\widehat{C}_{j_i}(r, \eta) R^\perp(r, \eta)) \right] \widehat{C}_{j_k}(r, \eta) \\ & \quad + \sum_{s=2}^n \sum_{\substack{n_1+\dots+n_s=n \\ n_i \geq 1}} \left[ \prod_{i=1}^{s-1} (\widehat{G}_{n_i}(r, \eta) R^\parallel(r, \eta)) \right] \widehat{G}_{n_s}(r, \eta), \end{aligned}$$

where on the last right hand side the first term originates from  $s = 1$  and the second term from summing over all  $s \geq 2$ . Collecting equalities yields the claim.  $\square$

## 4.5 Estimating the Renormalized Graphs

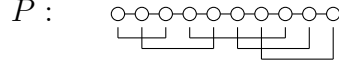
Next we will estimate the expression in the above formula. To this end we need a few lemmas.

**Lemma 11.** *Let  $X$  be a finite subset of  $\mathbb{Z}$  containing at least four elements. Let  $P$  be a linked pair partition of  $X$ , and let  $p \in P$ . Then there exists a pair  $q \in P$  different from  $p$  such that  $P \setminus \{q\}$  is again a linked pair partition of  $X \setminus q$ .*

*Example 9.* Consider the set  $X = \{x_1, x_2, \dots, x_{10}\} \subset \mathbb{Z}$ , with

$$x_1 < x_2 < \dots < x_{10},$$

which is indicated by the circles in the diagram below (where the index increases from left to right). We consider the pair partition  $P$  which is indicated by the lines below.



If we remove one of the pairs  $\{x_2, x_5\}$  or  $\{x_4, x_8\}$ , then the set of the remaining pairs is not linked anymore. If we remove one of the three pairs  $\{x_1, x_3\}$ ,  $\{x_6, x_9\}$ , or  $\{x_7, x_{10}\}$ , then the set of the remaining pairs is linked. Since we can remove any of the aforementioned three pairs, it follows that for any  $p \in P$  there exists a  $q \in P$  different from  $p$  such that  $P \setminus \{q\}$  is again a linked pair partition of  $X \setminus q$ .

*Proof.* For any two  $r, s \in P$  define the distance

$$d_P(r, s) := \inf\{l \in \mathbb{N} : \text{there exists a linked path in } P \text{ of length } l \text{ from } r \text{ to } s\}.$$

Clearly,  $d_P$  is a metric on  $P$ . Define

$$m_P(p) := \max\{d(p, r) : r \in P\}.$$

Since  $X$  is finite we can pick a  $q \in P$  such that

$$d_P(p, q) = m_P(p).$$

Then  $P \setminus \{q\}$  is again linked. Otherwise there would exist at least two linked components. One component must contain  $p$  and we could pick an  $r \in P \setminus \{q\}$  in a different component. But then every linked path in  $P$  from  $p$  to  $r$  would have to pass through  $q$ . This would imply  $d_P(p, q) < d_P(p, r) \leq m_P(p)$ . This is a contradiction to the choice of  $q$ .  $\square$

**Lemma 12.** *Let  $X \subset \mathbb{Z}$  be a finite set. Let  $P$  be a linked pair partition of  $X$ .*

(a) *Suppose we are given,*

$$\phi = ((V)_{x \in X}, (F_e)_{e \in E_X}),$$

*a  $(\mathcal{V}_{\text{sb}}, \mathcal{R}_{\text{sb}})$ -valued graph function on  $X$ . Suppose there exists a constant  $C_F$  such that for all  $e \in E_X$  we have*

$$\|F_e(r)\| \leq C_F(|r|^{-1} + 1), \forall r \geq 0.$$

*Then*

$$\sup_{r \geq 0} \|\mathcal{C}_P(\phi)(r)\| \leq C_F^{|X|} C_1^{|X|-2} C_0^2,$$

*where*

$$C_p := \left( \int dk (|k|^{-1} + 1)^{p+1} \|G(k)\|^2 \right)^{1/2}.$$

(b) Let  $S \subset \mathbb{R}^d$ . Suppose for each  $s \in S$  we are given

$$\phi_s = ((V)_{x \in X}, (F_{e,s})_{e \in E_X}),$$

a  $(\mathcal{V}_{\text{sb}}, \mathcal{R}_{\text{sb}})$ -valued graph function on  $X$ . Suppose for each  $r > 0$  and for each  $e \in E_X$ , the function  $s \mapsto F_{e,s}(r)$  is continuous, and suppose there exists a constant  $C_F$  such that

$$\|F_{e,s}(r)\| \leq C_F(|r|^{-1} + 1), \quad \forall r > 0, s \in S, e \in E_X.$$

Then the function  $s \mapsto \mathcal{C}_P(\phi_s)(r)$  is continuous for each  $r \geq 0$ .

*Proof.* (a) Since  $P$  is a pair partition of  $X$  the cardinality of  $X$  must be even. Thus we have  $|X| = 2n$  for some  $n \in \mathbb{N}$ . Using the notation introduced in Definition 5 we estimate

$$\begin{aligned} \|\mathcal{C}_P(\phi)(r)\| &\leq \int \prod_{x \in X} \{dk_x\} \delta_P(k) \\ &\quad \times \prod_{j \in X \setminus \{\max X\}} \left\{ \|G_{j,P}^\#(k_j)\| \|F_{\{j, r_X(j)\}}(r + |K_{\{j, r_X(j)\}}|_P)\| \right\} \|I_{\max X, P}(k_{\max X})\| \\ &\leq \sup_{\substack{P \text{ linked pair} \\ \text{partition of } X}} \int \prod_{x \in X} \{dk_x\} \delta_P(k) \\ &\quad \times \prod_{j \in X \setminus \{\max X\}} \left\{ \|G_{j,P}^\#(k_j)\| C_F(|K_{\{j, r_X(j)\}}|_P^{-1} + 1) \right\} \|I_{\max X, P}(k_{\max X})\| =: \text{Est}_n. \end{aligned} \tag{4.20}$$

We will show by induction in  $n$  that

$$\text{Est}_n \leq C_F^{2n-1} C_1^{2n-2} C_0^2.$$

First we consider the case  $n = 1$ .

$$\begin{aligned} \text{Est}_1 &= \int dk_{\min X} dk_{\max X} \delta(k_{\min X} - k_{\max X}) \|G^*(k_{\min X})\| C_F(|k_{\max X}|^{-1} + 1) \|G(k_{\max X})\| \\ &\leq C_F C_0^2. \end{aligned}$$

Next we show the induction step  $n - 1 \rightarrow n$ . The goal is to integrate out a pair of paired variables, such that the set of pairings of the remaining variables remains linked. The details, which are illustrated in the example below, are as follows. Let  $P$  be a linked pair partition of  $X$ , such that (4.20) is maximal. By Lemma 11, we can pick a pair  $q \in P$  such that  $P_q := P \setminus \{q\}$  is a linked pair partition of  $X_q := X \setminus q$ . We want to remove the propagators over the edges, which are adjacent to  $q$  and lie in the span of  $q$ , i.e. the edges

$$e_l := \{\min q, r_X(\min q)\}, \quad e_r := \{l_X(\max q), \max q\},$$

where we introduced the notation  $l_X(x) := \max(X \setminus [x, \infty))$  (denoting the nearest neighbor on the left of  $x$  in  $X$ ). To this end we will use the following lower bounds, which are an immediate consequence of the definition,

$$|K_{e_l}|_P \geq |k_{\max q}|, \quad |K_{e_r}|_P \geq |k_{\max q}|, \tag{4.21}$$

$$|K_e|_P \geq |K_e|_{P_q}, \quad \forall e \in E_X \cap E_{X_q}, \quad (4.22)$$

and

$$|K_{\{l_X(\min q), \min q\}}|_P \geq |K_{\{l_X(\min q), r_X(\min q)\}}|_{P_q}, \quad \text{provided } \min q \neq \min X, \quad (4.23)$$

$$|K_{\{q, r_X(\max q)\}}|_P \geq |K_{\{l_X(\max q), r_X(\max q)\}}|_{P_q}, \quad \text{provided } \max q \neq \max X. \quad (4.24)$$

We use the estimates in (4.21)–(4.24) to obtain an upper bound for (4.20), integrate out the variables  $k_{\max q}$  and  $k_{\min q}$ , which are paired by a delta function, and use the inequality

$$\int dk_{\max q} dk_{\min q} \delta(k_{\max q} - k_{\min q}) (|k_{\max q}|^{-1} + 1) (1 + |k_{\max q}|^{-1}) \|G(k_{\min q})\| \|G(k_{\max q})\| \leq C_1^2.$$

This yields the following estimate

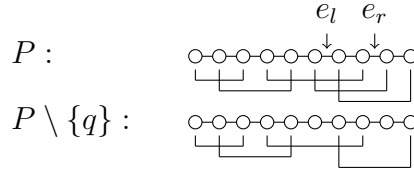
$$\begin{aligned} \text{Est}_n &\leq C_F^2 C_1^2 \int \prod_{x \in X_q} \{dk_x\} \delta_{P_q}(k) \\ &\quad \times \prod_{j \in X_q \setminus \{\max X_q\}} \left\{ \|I_{j, P_q}(k_j)\| C_F (|K_{\{j, r_{X_q}(j)\}}|_{P_q}^{-1} + 1) \right\} \|I_{\max X_q, P_q}(k_{\max X_q})\| \\ &\leq C_1^2 C_F^2 \text{Est}_{n-1}. \end{aligned}$$

This shows the induction step.

*Example 10.* Consider the situation as in Example 9 above. If we choose  $q = \{x_6, x_9\}$ , then  $P \setminus \{q\}$  is a linked pair partition of  $X \setminus \{q\}$ . In that case we want to remove the propagators over the edges

$$e_l = \{x_6, x_7\}, \quad e_r = \{x_8, x_9\}$$

and we integrate over the pair of variables  $k_{\min q} = k_{x_6}$  and  $k_{\max q} = k_{x_9}$ .



(b) This follows from dominated convergence and part (a). Explicitly we estimate for  $s, t \in S$  and all  $r \geq 0$

$$\begin{aligned} &\|\mathcal{C}_P(\phi_s)(r) - \mathcal{C}_P(\phi_t)(r)\| \\ &\leq \sum_{l \in X \setminus \{\max X\}} \int \prod_{x \in X} \{dk_x\} \delta_P(k) \\ &\quad \times \prod_{j \in X \setminus \{\max X\}} \left\{ \|G_{j, P}^\#(k_j)\| \|1_{j \leq l} F_{\{j, r_X(j)\}, s}(r + |K_{\{j, r_X(j)\}}|_P) - 1_{j \geq l} F_{\{j, r_X(j)\}, t}(r + |K_{\{j, r_X(j)\}}|_P)\| \right\} \\ &\quad \times \|I_{\max X, P}(k_{\max X})\|. \end{aligned}$$

The factor for  $l = j$  converges to zero. The integrand can be estimated by using the triangle inequality for the factor  $l = j$ . This results in an integrand which is an upper bound and which can be integrated by the proof of (b).  $\square$

We need an analogous estimate on the derivative.

**Lemma 13.** *Let  $X \subset \mathbb{Z}$  be a finite set and let  $S \subset \mathbb{R}^d$  be open. Suppose for each  $s \in S$  we are given*

$$\phi_s = ((V)_{x \in X}, (F_{e,s})_{e \in E_X}),$$

*a  $(\mathcal{V}_{\text{sb}}, \mathcal{R}_{\text{sb}})$ -valued graph function on  $X$ . Suppose there exist constants  $C_F$  such that for all  $e \in E_X$  we have*

$$\|F_{e,s}(r)\| \leq C_F(|r|^{-1} + 1), \quad r > 0.$$

*Suppose that for each  $r > 0$  and for each  $e \in E_X$ , the function  $s \mapsto F_{e,s}(r)$  has continuous partial derivatives with*

$$\|\partial_{s_i} F_{e,s}(r)\| \leq C_F(|r|^{-1} + 1)^2, \quad r > 0, \quad i = 1, \dots, d.$$

*Then for any linked pair partition  $P$  of  $X$  the function  $s \mapsto \mathcal{C}_P(\phi_s)(r)$  has for each  $r \geq 0$  continuous partial derivatives and*

$$\|\partial_{s_i} \mathcal{C}_P(\phi_s(r))\| \leq (|X| + 1) C_F^{|X|-1} C_1^{|X|}.$$

*Proof.* As in the proof of the previous lemma we can assume that  $|X| = 2n$  for some  $n \in \mathbb{N}$ . Using the well known arguments ensuring the interchange of differentiation and integration, we essentially need to show that the norm of the differentiated integrand can be estimated from above by an integrable function. (Strictly speaking, only the estimates given in the following proof will justify the existence of the partial derivative  $\partial_{s_i} \mathcal{C}_P(\phi_s(r))$ . Nevertheless for notational compactness we shall already write  $\partial_{s_i} \mathcal{C}_P(\phi_s(r))$  in (4.25) and (4.26), below.) We calculate the derivative using Leibnitz rule. First we show the estimate in case  $n = 1$ , in which case we find

$$\begin{aligned} \|\partial_{s_i} \mathcal{C}_P(\phi_s(r))\| &\leq \int dk_{\min X} dk_{\max X} \delta(k_{\min X} - k_{\max X}) \|G^*(k_{\min X})\| \\ &\quad \times C_F((|k_{\max X}| + r)^{-1} + 1)^2 \|G(k_{\max X})\| \\ &\leq C_F C_1^2. \end{aligned} \tag{4.25}$$

Let us now consider the case  $n \geq 2$ . Calculating the derivative using Leibnitz rule we find that for all  $r > 0$  and  $n \geq 1$  we have

$$\begin{aligned} \|\partial_{s_i} \mathcal{C}_P(\phi_s(r))\| &\leq \sum_{l \in X \setminus \{\max X\}} \int \prod_{x \in X} \{dk_x\} \delta_P(k) \\ &\quad \times \prod_{j \in X \setminus \{\max X\}} \left\{ \|G_{j,P}^\sharp(k_j)\| C_F(|K_{\{j,r_X(j)\}}|_P^{-1} + 1)^{1+\delta_{jl}} \right\} \|I_{\max X, P}(k_{\max X})\| \\ &\leq (|X| - 1) \text{DEst}_n, \end{aligned} \tag{4.26}$$

where we defined

$$\begin{aligned} \text{DEst}_n &:= \sup_{l \in X \setminus \{\max X\}} \sup_{\substack{P \text{ linked} \\ \text{pair partition of } X}} \int \prod_{x \in X} \{dk_x\} \delta_P(k) \\ &\quad \times \prod_{j \in X \setminus \{\max X\}} \left\{ \|G_{j,P}^\sharp(k_j)\| C_F(|K_{\{j,r_X(j)\}}|_P^{-1} + 1)^{1+\delta_{jl}} \right\} \|G_{\max X, P}^\sharp(k_{\max X})\|. \end{aligned} \tag{4.27}$$

We want to show by induction that for all  $n \in \mathbb{N}$  we have

$$\text{DEst}_n \leq C_F^{2n-1} C_1^{2n}.$$

By (4.25) we know that the inequality for  $\text{DEst}_1$  holds. Next we show that  $n-1 \rightarrow n$ . Let us first sketch the idea. As in the proof of Lemma 12, we want to remove two propagators by integrating out a pair of paired variables, such that the set of pairings of the remaining variables is again linked, in addition, we want the term which contains a two in the exponent to remain in the integral. The details are as follows, and illustrated in the examples below. Suppose the maximum in (4.27) is attained for  $l \in X \setminus \{\max X\}$  and the linked pairing  $P$ . Consider the edge  $e := \{l, r_X(l)\}$  (for which we have a two in the exponent). We pick a  $p \in P$  such that  $e$  lies in the span of  $p$  (this can always be achieved, since  $P$  is linked and therefore connected). By Lemma 11 there exists an element  $q \in P \setminus \{p\}$  such that  $P_q := P \setminus \{q\}$  is a linked pair partition of  $X_q = X \setminus q$ . If none of the edges

$$e_l := \{\min q, r_X(\min q)\}, \quad e_r := \{l_X(\max q), \max q\},$$

is equal to  $e$ , then an estimate as in the proof of Lemma 12 yields

$$\text{DEst}_n \leq C_1^2 C_F^2 \text{DEst}_{n-1}, \quad (4.28)$$

since the term involving  $e$  is not integrated out. If  $e = e_l$  (the case  $e = e_r$  is analogous) then we use the estimate

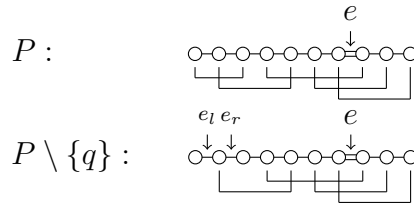
$$(|K_e|_P^{-1} + 1)^2 \leq (|K_{\{l_X(\min q), r_X(\min q)\}}|_{P_q}^{-1} + 1)(|k_{\max q}|^{-1} + 1).$$

The second term on the right hand side is again estimated as in the proof of Lemma 12, whereas the first term remains in the integral. This yields again (4.28). The continuity of the derivative follows from dominated convergence as in the proof of Lemma 12.

*Example 11.* Consider the situation as in Example 9 above. Suppose  $l = x_6$  and so  $e = \{x_6, x_7\}$ . Then for  $p = \{x_6, x_9\} \in P$  the span of  $p$  contains  $e$ . If we choose  $q = \{x_1, x_3\}$ , then  $P \setminus \{q\}$  is a linked pair partition of  $X \setminus \{q\}$ . In that case we want to remove the propagators over the edges

$$e_l = \{x_1, x_2\}, \quad e_r = \{x_2, x_3\},$$

which are both different from  $e$ .



*Example 12.* Consider the set  $X = \{x_1, x_2, x_3, x_4\}$ , where  $x_1 < x_2 < \dots < x_4$ . Let  $P$  be a linked pairing with pairs indicated in the diagram below. If  $l = x_2$ , then  $e = \{x_2, x_3\}$  and  $p = \{x_1, x_3\} \in P$  contains  $e$  in its span. In that case we can remove  $q = \{x_1, x_2\}$  and

$$e_l = \{x_1, x_2\}, \quad e_r = \{x_2, x_3\},$$

where  $e = e_r$ .

$$\begin{array}{ll}
P : & \begin{array}{c} e \\ \downarrow \\ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \\ \uparrow \quad \uparrow \quad \uparrow \\ e_l \quad e_r \end{array} \\
P \setminus \{q\} : & \begin{array}{c} \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \\ \uparrow \quad \uparrow \quad \uparrow \\ e_l \quad e_r \end{array}
\end{array}$$

□

*Proof of Theorem 1.* We prove the theorem by induction in  $n \in \mathbb{N}$ . We make the following induction hypothesis.

$I_n$ : There are unique numbers  $\mathcal{E}_m$  for  $m \in N_n$  such that the following holds for the functions  $C_m, G_m : [0, \infty) \times (0, 1] \rightarrow \mathcal{L}(\mathcal{H}_{\text{at}})$  defined in (4.8) and (4.17).

- (i) For  $m \in N_n$  the functions  $C_m, G_m$  are continuous on  $[0, \infty) \times (0, 1]$  and bounded and extend to continuous functions on  $[0, \infty) \times [0, 1]$ .
- (ii) For  $m \in N_n$  the functions  $C_m, G_m$  are on  $(0, \infty) \times (0, 1)$  continuously differentiable with respect to  $r$  and  $\eta$  with uniformly bounded derivatives.
- (iii) For  $m \in N_n$  we have  $\mathcal{E}_m = \langle \varphi_{\text{at}}, G_m(0, 0) \varphi_{\text{at}} \rangle$ .

First observe that by definition of  $C_n$  and  $G_n$  vanish for  $n$  odd. Moreover, we note that  $R^\perp(r, \eta)$  is continuous on  $[0, \infty) \times [0, 1]$  whereas  $R^\parallel(r, \eta)$  is continuous on  $(0, \infty) \times [0, 1]$ , with a discontinuity at  $r = 0$ .

For  $n = 2$ , the Hypothesis  $I_n$  can be seen as follows. We have by definition for  $r \geq 0$  and  $\eta > 0$

$$\begin{aligned}
C_2(r, \eta) &= \int G^*(k) \frac{1 - P_{\text{at}} 1_{|k|+r=0}}{H_{\text{at}} - E_{\text{at}} + |k| + \eta + r} G(k) dk \\
&= \int G^*(k) \frac{1}{H_{\text{at}} - E_{\text{at}} + |k| + \eta + r} G(k) dk,
\end{aligned}$$

where in the second equality we used that  $\{k \in \mathbb{R}^3 : |k| = 0\}$  is a set of measure zero. Note that we have  $G_2 = C_2$ . (i) follows from dominated convergence (or Lemma 12). (ii) follows from the usual results about interchanging integration and differentiation (or Lemma 13). (iii) follows from the definition  $\mathcal{E}_2 := \langle \varphi_{\text{at}}, G_2(0, 0) \varphi_{\text{at}} \rangle$ .

Now let us show the induction step. Suppose that  $I_n$  holds. If  $n$  is even, the induction hypothesis trivially holds for  $n + 1$  since in that case  $C_{n+1}, G_{n+1}$  vanish identically as a direct consequence of the definition. Thus suppose  $n$  is odd. By estimating the remainder of a first

order Taylor expansion, it follows from the induction hypothesis that for  $m \in N_n$  there exists constants  $d_m$  such that

$$|P_{\text{at}}\widehat{G}_m(r, \eta)P_{\text{at}}| \leq d_m|r + \eta|. \quad (4.29)$$

Next we observe that

$$\|R^\perp(r, \eta)\| \leq \frac{1}{\inf(\sigma(H_{\text{at}}) \setminus \{E_0\}) - E_0}, \quad (4.30a)$$

$$\|R^\parallel(r, \eta)\| \leq (r + \eta)^{-1}. \quad (4.30b)$$

From the induction hypothesis (i), Eq. (4.19) of Theorem 3 and Eqns. (4.29) and (4.30b), we see that for all  $m \in N_n$  there exists a constant  $c_m$  such that for all  $r \geq 0, \eta > 0$  we have

$$\|P_{\text{at}}\widehat{T}_m(r, \eta)P_{\text{at}}\| \leq c_m(r + \eta), \quad (4.31)$$

$$\|\bar{P}_{\text{at}}\widehat{T}_m(r, \eta)P_{\text{at}}\| \leq c_m, \quad (4.32)$$

$$\|P_{\text{at}}\widehat{T}_m(r, \eta)\bar{P}_{\text{at}}\| \leq c_m, \quad (4.33)$$

$$\|\bar{P}_{\text{at}}\widehat{T}_m(r, \eta)\bar{P}_{\text{at}}\| \leq c_m(r + \eta)^{-1}. \quad (4.34)$$

Now using the decomposition of the resolvent (4.16) and the bounds (4.31)–(4.34) and (4.30a) we see that for  $m \in N_n$  there exists a constant  $c_m$  such that

$$\|R(r, \eta)\widehat{T}_m(r, \eta)R(r, \eta)\| \leq \frac{c_m}{r + \eta}.$$

Moreover, we see from (4.19) and the induction hypothesis (i) that  $R(r, \eta)\widehat{T}_m(r, \eta)R(r, \eta)$  is continuous on  $(0, \infty) \times (0, 1]$  and has a continuous extension to  $(0, \infty) \times [0, 1]$ . Thus it follows from the definition of  $C_n$ , given in (4.8), and Lemma 12 that  $C_{n+1}$  is bounded and has a continuous extension to  $[0, \infty) \times [0, 1]$ . Now it follows from (4.17) that the same holds for  $G_{n+1}$ . Thus we have shown (i) for  $n + 1$ .

From (4.19) of Theorem 3 and the induction hypothesis (ii) we see that  $\widehat{T}_n$  is continuously differentiable on  $(0, \infty) \times (0, 1)$ . Now let  $\xi = r$  or  $\xi = \eta$ . Calculating the derivative using the product rule, we obtain similarly as before, with Eq. (4.29) and

$$\|\partial_\xi R^\perp(r, \eta)\| \leq \frac{1}{(\inf(\sigma(H_{\text{at}}) \setminus \{E_0\}) - E_0)^2}, \quad \|\partial_\xi R^\parallel(r, \eta)\| \leq (r + \eta)^{-2},$$

the bounds

$$\|P_{\text{at}}\partial_\xi\widehat{T}_n(r, \eta)P_{\text{at}}\| \leq c_m, \quad (4.35)$$

$$\|\bar{P}_{\text{at}}\partial_\xi\widehat{T}_n(r, \eta)P_{\text{at}}\| \leq c_m(r + \eta)^{-1}, \quad (4.36)$$

$$\|P_{\text{at}}\partial_\xi\widehat{T}_n(r, \eta)\bar{P}_{\text{at}}\| \leq c_m(r + \eta)^{-1}, \quad (4.37)$$

$$\|\bar{P}_{\text{at}}\partial_\xi\widehat{T}_n(r, \eta)\bar{P}_{\text{at}}\| \leq c_m(r + \eta)^{-2}. \quad (4.38)$$

Now using (4.35)–(4.38) we obtain for  $r > 0$  and  $\eta > 0$  the bound

$$\|\partial_\xi R(r, \eta)\widehat{T}_n(r, \eta)R(r, \eta)\| \leq \frac{C}{(r + \eta)^2}.$$



Now we see from the definition of  $C_n$ , (4.8), and Lemma 13, that  $C_{n+1}$  is continuously differentiable on  $(0, \infty) \times (0, 1)$  with uniformly bounded derivatives. Hence it follows from (4.17) that the same holds for  $G_{n+1}$ . Thus we have shown (ii) for  $n + 1$ . Property (iii) now follows from the definition  $\mathcal{E}_{n+1} := \langle \varphi_{\text{at}}, G_{n+1}(0, 0) \varphi_{\text{at}} \rangle$ . Thus we have shown  $I_{n+1}$ .

Using (4.18) of Theorem 3, it follows from (4.29) and (4.30a) that

$$\lim_{\eta \downarrow 0} \langle \varphi_{\text{at}}, T_n(0, \eta) \varphi_{\text{at}} \rangle = \langle \varphi_{\text{at}}, G_n(0, 0) \varphi_{\text{at}} \rangle = \mathcal{E}_n,$$

where the last equality follows from (iii) of the induction hypothesis. Setting  $E_n := \mathcal{E}_n$ , the claim of the theorem follows from Lemma 6.  $\square$

As a byproduct of the proof we obtain the following corollary, which tells us that we can calculate the coefficients  $E_n$  solely in terms of linked pair partitions.

**Corollary 1.** *Let the situation be as in Theorem 1. Then we have*

$$E_n = \lim_{\eta \downarrow 0} \langle \varphi_{\text{at}}, \left\{ C_n(0, \eta) + \sum_{k=2}^n \sum_{\substack{j_1 + \dots + j_k = n \\ j_i \geq 1}} \left[ \prod_{i=1}^{k-1} (\widehat{C}_{j_i}(0, \eta) R^\perp(0, \eta)) \right] \widehat{C}_{j_k}(0, \eta) \right\} \varphi_{\text{at}} \rangle.$$

## 5 Ground State

**Theorem 4.** Suppose the assumptions of Theorem 1 hold and let  $(E_n)_{n \in \mathbb{N}}$  be the unique sequence given in Theorem 1. Let

$$\psi_0 = \varphi_{\text{at}} \otimes \Omega.$$

Then for all  $m \in \mathbb{N}$  the following limit exists

$$\psi_m = \lim_{\eta \downarrow 0} \psi_m(\eta),$$

where

$$\psi_m(\eta) := \sum_{k=1}^m \sum_{\substack{j_1 + \dots + j_k = m \\ j_s \geq 1}} \prod_{s=1}^k \{ (E_0 - H_0 - \eta)^{-1} \bar{P}_0(\delta_{1j_s} V - E_{j_s}) \} \psi_0. \quad (5.1)$$

To show that the expansion coefficients of the ground state exist, we have to calculate their norm. For this we introduce the following graph functions. For  $m, n \in \mathbb{Z}$  with  $m \leq n$  we define the set

$$N_{m,n} = [m, n] \cap \mathbb{Z} \setminus \{0\},$$

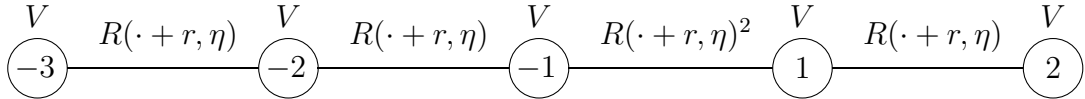
and

$$\pi_{m,n}(r, \eta) = ((V_x)_{x \in N_{m,n}}, (R_e(\cdot + r, \eta))_{e \in E_{N_{m,n}}}),$$

where for  $e \in E_{N_n}$  we defined

$$R_e(r, \eta) = \begin{cases} R(r, \eta)^2, & \text{if } e = \{-1, 1\} \\ R(r, \eta), & \text{otherwise.} \end{cases}$$

*Example 13.* We can write  $\pi_{-3,2}(r, \eta)$  symbolically as



Note that for  $n \in \mathbb{N}$  we have  $\pi_{1,n}(r, \eta) = \pi_n(r, \eta)$ . For  $M \subset \mathbb{Z}$  we define the set  $\mathcal{Q}_0(M)$  consisting of all collections of disjoint nonempty intervals of  $M$ , such that 0 does not lie between the endpoints of any of the intervals, i.e.,

$$\begin{aligned} \mathcal{Q}_0(M) &:= \{ \mathcal{I} \subset \mathcal{P}(M) : \forall I, J \in \mathcal{I} \text{ we have } I \cap J = \emptyset, \\ &\quad \text{If } I \in \mathcal{I}, \text{ then } 0 \notin [\min I, \max I], \\ &\quad \forall I \in \mathcal{I} \text{ the set } I \text{ is a nonempty interval of } M \}. \end{aligned}$$

Note that  $\mathcal{Q}_0(M) \subset \mathcal{Q}(M)$ . We define

$$\begin{aligned} T_{m,n}(r, \eta) &:= P_\Omega \Pi(\pi_{m,n}(r, \eta)) P_\Omega + \sum_{\substack{\mathcal{I} \in \mathcal{Q}_0(N_{m,n}): \\ |\mathcal{I}| \geq 1}} P_\Omega \Pi(\text{subst}_{\substack{I \rightarrow -E_{|I|} \\ I \in \mathcal{I}}}(\pi_{m,n}(r, \eta))) P_\Omega \\ &= \sum_{\mathcal{I} \in \mathcal{Q}_0(N_{m,n})} P_\Omega \Pi(\text{subst}_{\substack{I \rightarrow -E_{|I|} \\ I \in \mathcal{I}}}(\pi_{m,n}(r, \eta))) P_\Omega, \end{aligned} \quad (5.2)$$

as an operator on the atomic Hilbert space. As an immediate consequence of the definitions we obtain the following lemma. To be explicit we give a proof below.

**Lemma 14.** *Suppose the assumptions of Theorem 4 hold. Then with the definition (5.1) we have for all  $m \in \mathbb{N}$  that*

$$\langle \psi_m(\eta), \psi_m(\eta) \rangle = \langle \varphi_{\text{at}}, T_{-m,m}(0, \eta) \varphi_{\text{at}} \rangle.$$

*Proof.* The proof is analogous to the proof of Lemma 6. Inserting (5.1) into the left hand side and taking the adjoint we find

$$\begin{aligned} \langle \psi_m(\eta), \psi_m(\eta) \rangle &= \sum_{k'=1}^m \sum_{k=1}^m \sum_{\substack{j'_1 + \dots + j'_{k'} = m \\ j'_s \geq 1}} \sum_{\substack{j_1 + \dots + j_k = m \\ j_s \geq 1}} \\ &\langle \psi_0, \prod_{s'=1}^{k'} \left\{ (\delta_{1j'_{s'}} V - E_{j'_{s'}}) (E_0 - H_0 - \eta)^{-1} \bar{P}_0 \right\} \prod_{s=1}^k \left\{ (E_0 - H_0 - \eta)^{-1} \bar{P}_0 (\delta_{1j_s} V - E_{j_s}) \right\} \psi_0 \rangle. \end{aligned} \quad (5.3)$$

Consider the summand in (5.2) indexed by  $\mathcal{I} \in \mathcal{Q}_0(N_{-m,m})$ . We partition the set  $\mathcal{I}$  into  $\mathcal{I}_1 = \{I \in \mathcal{I} : I \subset N_{-m,-1}\}$  and  $\mathcal{I}_2 = \{I \in \mathcal{I} : I \subset N_{1,m}\}$ . By definition of  $\mathcal{Q}_0(N_{-m,m})$  this is indeed a partition of  $\mathcal{I}$ . As in the proof of Lemma 6 we define

$$\begin{aligned} \mathcal{K}_1 &:= \{\{s\} : s \in N_{-m,-1} \text{ and } s \notin I = \emptyset, \forall I \in \mathcal{I}_1\} \\ \mathcal{K}_2 &:= \{\{s\} : s \in N_{1,m} \text{ and } s \notin I = \emptyset, \forall I \in \mathcal{I}_2\}. \end{aligned}$$

Now we order the elements of  $\mathcal{S}_j := \mathcal{I}_j \cup \mathcal{K}_j$  in increasing order as in the proof of Lemma 6. This defines a bijection  $\varphi_j : N_{|\mathcal{S}_j|} \rightarrow \mathcal{S}_j$  preserving the order. By construction we see that the summand in (5.2) indexed by  $\mathcal{I}$  is equal to the summand in (5.3) which we obtain by choosing  $k' = |\mathcal{S}_1|$  and  $k = |\mathcal{S}_2|$ ,  $j'_{s'} = |\varphi_1(s')|$  and  $j_s = |\varphi_2(s)|$ , by choosing  $-E_1$  in case  $j'_{s'} = 1$  and  $\varphi_1(s') \in \mathcal{I}_1$  or  $j_s = 1$  and  $\varphi_2(s) \in \mathcal{I}_2$ , and by choosing  $V$  in case  $j'_{s'} = 1$  and  $\varphi_1(s') \notin \mathcal{I}_1$  or  $j_s = 1$  and  $\varphi_2(s) \notin \mathcal{I}_2$ .  $\square$

For  $m, n \in \mathbb{Z}$  with  $m \leq n$  we define

$$C_{m,n}(r, \eta) := \sum_{\substack{P_e \text{ linked pairing of } N_{m,n} \\ S(P_e)=[m,n]}} \mathcal{C}_{P_e}^0 \left( \text{subst}_{\substack{I \rightarrow \tilde{T}_I(+r, \eta) \\ I \in \mathcal{I}_{P_e}}} (\pi_{m,n}(r, \eta)) \right)$$

where

$$\tilde{T}_I(r, \eta) := \begin{cases} T_{\min I, \max I}(r, \eta) - E_{|I|+1}, & \text{if } 0 \notin [\min I, \max I] \\ T_{\min I, \max I}(r, \eta), & \text{otherwise.} \end{cases}$$

Observe that if  $m, n \in \mathbb{Z}$  have the same sign and  $m \leq n$ , then

$$T_{m,n} = T_{n-m+1}, \quad C_{m,n} = C_{n-m+1}.$$

**Proposition 2.** *For  $m, n \in \mathbb{Z}$  with  $m \leq n$  we have*

$$\begin{aligned} T_{m,n}(r, \eta) &= C_{m,n}(r, \eta) \\ &+ \sum_{k=2}^{n-m} \sum_{\substack{j_1 + \dots + j_k = n-m \\ j_i \geq 1}} \left[ \prod_{i=1}^{k-1} (\tilde{C}_{l_i(m, \underline{j}), r_i(m, \underline{j})}(r, \eta) \tilde{R}_{\{r_i(m, \underline{j}), l_{i+1}(m, \underline{j})\}}(r, \eta)) \right] \\ &\times \tilde{C}_{l_k(m, \underline{j}), r_k(m, \underline{j})}(r, \eta), \end{aligned} \quad (5.4)$$

where we defined inductively for  $\underline{j} = (j_1, \dots, j_k)$  the numbers  $l_1(m, \underline{j}) := m$  and  $l_{i+1}(m, \underline{j}) := r_{N_{m,n}}^{j_i}(l_i(m, \underline{j}))$ , and we defined  $r_i(m, \underline{j}) := r_{N_{m,n}}^{j_{i-1}}(l_i(m, \underline{j}))$  and

$$\tilde{C}_{p,q}(r, \eta) := \begin{cases} C_{p,q}(r, \eta), & \text{if } 0 \in [p, q] \\ C_{p,q}(r, \eta) - E_{q-p+1}, & \text{otherwise} \end{cases}.$$

The proof is very similar to that of Proposition 1, except we have to consider the factor involving the square of the resolvent and the fact that we have less energy subtractions.

*Proof.* The case, where  $m, n$  have the same sign, has already been shown in the last section. Thus assume  $m < 0 < n$ . By the generalized Wick theorem, Lemma 9, we have

$$\begin{aligned} T_{m,n}(r, \eta) &= \sum_{P \text{ pairing of } N_{m,n}} \sum_{\mathcal{I} \in \mathcal{Q}_0(N_{m,n})} \mathcal{C}_P^0(\text{subst}_{I \rightarrow -\mathcal{E}_{|I|}}(\pi_{m,n}(r, \eta))) \\ &= T_{m,n}^{(C)}(r, \eta) + T_{m,n}^{(D)}(r, \eta), \end{aligned}$$

where we divided the sum over the partitions into partitions which connect the smallest and the largest vertex

$$T_{m,n}^{(C)}(r, \eta) := \sum_{\substack{P \text{ pairing of } N_{m,n} \\ S(P)=[m,n]}} \sum_{\mathcal{I} \in \mathcal{Q}_0(N_{m,n})} \mathcal{C}_P^0(\text{subst}_{I \rightarrow -\mathcal{E}_{|I|}}(\pi_{m,n}(r, \eta)))$$

and partitions which are disconnected

$$T_{m,n}^{(D)}(r, \eta) := \sum_{\substack{P \text{ pairing of } N_{m,n} \\ S(P) \neq [m,n]}} \sum_{\mathcal{I} \in \mathcal{Q}_0(N_{m,n})} \mathcal{C}_P^0(\text{subst}_{I \rightarrow -\mathcal{E}_{|I|}}(\pi_{m,n}(r, \eta)))$$

To simplify the connected part  $T_n^{(C)}(r, \eta)$  we use the decomposition (4.10) as in the proof of Proposition 1 and an analogous argument yields

$$\begin{aligned} &T_{m,n}^{(C)}(r, \eta) \\ &= \sum_{\substack{P_e \text{ pairing of } N_{m,n} \\ \{1,n\} \subset \bigcup P_e \\ P_e \text{ linked}}} \prod_{I \in \mathcal{I}_{P_e}} \left\{ \sum_{P_I \text{ pairing of } I} \right\} \sum_{\mathcal{I} \in \mathcal{Q}_0(N_{m,n})} \mathcal{C}_{(P_e \cup \bigcup_{I \in \mathcal{I}_{P_e}} P_I)}^0(\text{subst}_{J \rightarrow -\mathcal{E}_{|J|}}(\pi_{m,n}(r, \eta))) \\ &= \sum_{\substack{P_e \text{ pairing of } N_n \\ \{1,n\} \subset \bigcup P_e \\ P_e \text{ linked}}} \prod_{I \in \mathcal{I}_{P_e}} \left\{ \sum_{\mathcal{I}_I \in \mathcal{Q}_0(I)} \sum_{\substack{P_I \text{ pairing} \\ \text{of } I \setminus \bigcup \mathcal{I}_I}} \right\} \mathcal{C}_{(P_e \cup \bigcup_{I \in \mathcal{I}_{P_e}} P_I)}^0(\text{subst}_{J \rightarrow -\mathcal{E}_{|J|}}(\pi_{m,n}(r, \eta))) \\ &= \sum_{\substack{P_e \text{ pairing of } N_n \\ \{1,n\} \subset \bigcup P_e \\ P_e \text{ linked}}} \mathcal{C}_{P_e}^0(\text{subst}_{I \rightarrow \tilde{T}_I(\cdot, r, \eta)}(\pi_{m,n}(r, \eta))) \\ &= C_{m,n}(r, \eta). \end{aligned}$$

To simplify the disconnected part  $T_{m,n}^{(D)}(r, \eta)$  we rearrange the sum as in the proof of Proposition 1, which yields the following identities,

$$\begin{aligned}
T_{m,n}^{(D)}(r, \eta) &= \sum_{\substack{\mathcal{K} \in \mathcal{Q}(N_{m,n}) \\ \bigcup \mathcal{K} = N_{m,n} \\ |\mathcal{K}| \geq 2}} \prod_{K \in \mathcal{K}} \left\{ \sum_{\substack{P_K \text{ pairing of } K \\ S(P_K) = [\min K, \max K]}} \sum_{\mathcal{I}_K \in \mathcal{Q}_0(K)} + 1_{(P_K = \emptyset, \mathcal{I}_K = \{K\}, 0 \notin [\min K, \max K])} \right\} \\
&\quad \times \mathcal{C}_{\left(\bigcup_{K \in \mathcal{K}} P_K\right)}^0 \left( \text{subst}_{\substack{I \rightarrow -\mathcal{E}_{|I|} \\ I \in \bigcup_{K \in \mathcal{K}} \mathcal{I}_K}} (\pi_{m,n}(r, \eta)) \right) \\
&= \sum_{\substack{\mathcal{K} \in \mathcal{Q}(N_{m,n}) \\ \bigcup \mathcal{K} = N_{m,n} \\ |\mathcal{K}| \geq 2}} \prod_{K \in \mathcal{K}} \left\{ \sum_{\substack{P_K \text{ pairing of } K \\ S(P_K) = [\min K, \max K]}} \sum_{\mathcal{I}_K \in \mathcal{Q}_0(K)} + 1_{(P_K = \emptyset, \mathcal{I}_K = \{K\}, 0 \notin [\min K, \max K])} \right\} \\
&\quad \times \prod_{K \in \mathcal{K} \setminus \max \mathcal{K}} \left\{ \mathcal{C}_{P_K}^0 \left( \text{subst}_{\substack{I \rightarrow -\mathcal{E}_{|I|} \\ I \in \mathcal{I}_K}} (\pi_{\min K, \max K}(r, \eta)) \right) [R(r, \eta)]^{1+1_{0 \in [\min K, \max K]}} \right\} \\
&\quad \times \mathcal{C}_{P_{\max \mathcal{K}}}^0 \left( \text{subst}_{\substack{I \rightarrow -\mathcal{E}_{|I|} \\ I \in \mathcal{I}_{\max \mathcal{K}}}} (\pi_{\min(\max \mathcal{K}), \max(\max \mathcal{K})}(r, \eta)) \right) \\
&= \sum_{k=2}^{n-m} \sum_{\substack{j_1 + \dots + j_k = n-m \\ j_i \geq 1}} \left[ \prod_{i=1}^{k-1} (\tilde{C}_{l_i(m, \underline{j}), r_i(m, \underline{j})}(r, \eta) \tilde{R}_{\{r_i(m, \underline{j}), l_{i+1}(m, \underline{j})\}}(r, \eta)) \right] \tilde{C}_{l_k(m, \underline{j}), r_k(m, \underline{j})}(r, \eta),
\end{aligned}$$

where we ordered  $\mathcal{K}$  with respect to the ordering defined in (4.15), and in the last equality we identified the summation indices as follows:  $k = |\mathcal{K}|$  and  $j_i = |K_i|$ , for  $\mathcal{K} = \{K_1, K_2, \dots, K_k\}$  with  $K_1 < K_2 < \dots < K_k$ .  $\square$

As in Section 4.4, we want to decompose the resolvent occurring in (5.4) according to (4.16). To this end we define

$$\begin{aligned}
G_{m,n}(r, \eta) &:= C_{m,n}(r, \eta) \\
&\quad + \sum_{k=2}^{n-m} \sum_{\substack{j_1 + \dots + j_k = n-m \\ j_i \geq 1}} \left[ \prod_{i=1}^{k-1} (\tilde{C}_{l_i(m, \underline{j}), r_i(m, \underline{j})}(r, \eta) P^\perp \tilde{R}_{\{r_i(m, \underline{j}), l_{i+1}(m, \underline{j})\}}(r, \eta)) \right] \\
&\quad \times \tilde{C}_{l_k(m, \underline{j}), r_k(m, \underline{j})}(r, \eta), \tag{5.5}
\end{aligned}$$

where

$$P^\perp := 1 - P_{\text{at}}.$$

Moreover, we define

$$\tilde{G}_{p,q}(r, \eta) := \begin{cases} G_{p,q}(r, \eta), & \text{if } 0 \in [p, q] \\ G_{p,q}(r, \eta) - E_{q-p+1}, & \text{otherwise.} \end{cases}$$

and

$$P^\parallel := P_{\text{at}}.$$

**Theorem 5.** *We have*

$$\begin{aligned}
T_{m,n}(r, \eta) &= G_{m,n}(r, \eta) \\
&+ \sum_{k=2}^{n-m} \sum_{\substack{j_1+\dots+j_k=n-m \\ j_i \geq 1}} \left[ \prod_{i=1}^{k-1} (\tilde{G}_{l_i(m, \underline{j}), r_i(m, \underline{j})}(r, \eta) P^{\parallel} \tilde{R}_{\{r_i(m, \underline{j}), l_{i+1}(m, \underline{j})\}}(r, \eta)) \right] \\
&\times \tilde{G}_{l_k(m, \underline{j}), r_k(m, \underline{j})}(r, \eta). \tag{5.6}
\end{aligned}$$

*Proof.* We start with the formula in Proposition 2 and write the resolvent as a sum of parallel and orthogonal part. Then we multiply out the resulting expression and, as in the proof of Theorem 3, we collect the terms according to the number,  $s-1$ , of times  $P^{\parallel}$  occurs. Starting with the second term of the right hand side of (5.4) we obtain by straight forward algebraic calculation

$$\begin{aligned}
&\sum_{k=2}^{n-m} \sum_{\substack{j_1+\dots+j_k=n-m \\ j_i \geq 1}} \left[ \prod_{i=1}^{k-1} (\tilde{C}_{l_i(m, \underline{j}), r_i(m, \underline{j})}(r, \eta) \tilde{R}_{\{r_i(m, \underline{j}), l_{i+1}(m, \underline{j})\}}(r, \eta)) \right] \tilde{C}_{l_k(m, \underline{j}), r_k(m, \underline{j})}(r, \eta) \\
&= \sum_{k=2}^{n-m} \sum_{\sigma_1, \dots, \sigma_{k-1} \in \{\perp, \parallel\}} \sum_{\substack{j_1+\dots+j_k=n-m \\ j_i \geq 1}} \\
&\quad \left[ \prod_{i=1}^{k-1} (\tilde{C}_{l_i(m, \underline{j}), r_i(m, \underline{j})}(r, \eta) P^{\sigma_i} \tilde{R}_{\{r_i(m, \underline{j}), l_{i+1}(m, \underline{j})\}}(r, \eta)) \right] \\
&\quad \times \tilde{C}_{l_k(m, \underline{j}), r_k(m, \underline{j})}(r, \eta) \\
&= \sum_{s=1}^{n-m} \sum_{\substack{n_1+\dots+n_s=n-m \\ n_i \geq 1}} \sum_{\substack{k_1, \dots, k_s \in \mathbb{N} \\ k_1+\dots+k_s \geq 2}} \sum_{\substack{j_{1,1}+\dots+j_{1,k_1}=n_1 \\ j_{1,i} \geq 1}} \dots \sum_{\substack{j_{s,1}+\dots+j_{s,k_s}=n_s \\ j_{s,i} \geq 1}} \\
&\quad \left[ \prod_{i=1}^{k_1-1} (\tilde{C}_{l_{i_1}(m, \underline{j}_1), r_{i_1}(m, \underline{j}_1)} P^{\perp} \tilde{R}_{\{r_{i_1}(m, \underline{j}_1), l_{i_1+1}(m, \underline{j}_1)\}}(r, \eta)) \right] \\
&\quad \times \tilde{C}_{l_{k_1}(m, \underline{j}_1), r_{k_1}(m, \underline{j}_1)} P^{\parallel} \tilde{R}_{r_{k_1}(m, \underline{j}_1), l_1(m, \underline{n}), \underline{j}_2)} \\
&\quad \dots \\
&\quad \left[ \prod_{i=1}^{k_s-1} (\tilde{C}_{l_{i_s}(l_{s-1}(m, \underline{n}), \underline{j}_s), r_{i_s}(l_{s-1}(m, \underline{n}), \underline{j}_s)} P^{\perp} \tilde{R}_{\{r_{i_s}(l_{s-1}(m, \underline{n}), \underline{j}_s), l_{i_s+1}(l_{s-1}(m, \underline{n}), \underline{j}_s)\}}(r, \eta)) \right] \\
&\quad \times \tilde{C}_{l_{k_s}(l_{s-1}(m, \underline{n}), \underline{j}_s), r_{k_s}(l_{s-1}(m, \underline{n}), \underline{j}_s)} \\
&= \sum_{k=2}^{n-m} \sum_{\substack{j_1+\dots+j_k=n-m \\ j_i \geq 1}} \left[ \prod_{i=1}^{k-1} (\tilde{C}_{l_i(m, \underline{j}), r_i(m, \underline{j})}(r, \eta) P^{\perp} \tilde{R}_{\{r_i(m, \underline{j}), l_{i+1}(m, \underline{j})\}}(r, \eta)) \right] \tilde{C}_{l_k(m, \underline{j}), r_k(m, \underline{j})}(r, \eta) \\
&+ \sum_{s=2}^{n-m} \sum_{\substack{n_1+\dots+n_s=n-m \\ n_i \geq 1}} \left[ \prod_{i=1}^{s-1} (\tilde{G}_{l_i(m, \underline{n}), r_i(m, \underline{n})}(r, \eta) P^{\parallel} \tilde{R}_{\{r_i(m, \underline{n}), l_{i+1}(m, \underline{n})\}}(r, \eta)) \right] \tilde{G}_{l_s(m, \underline{n}), r_k(m, \underline{n})}(r, \eta), \tag{5.7}
\end{aligned}$$

where the first term on the very right hand side, originates from the case  $s = 1$ , and for the second term we used the definition in (5.5). Adding  $C_{m,n}(r, \eta)$  to the right hand side of (5.7) and using again the definition in (5.5), the theorem follows.  $\square$

**Lemma 15.** *Let  $X \subset \mathbb{Z}$  be a finite set and let  $S \subset \mathbb{R}^d$ . Suppose for each  $s \in S$  we are given*

$$\phi_s = ((V)_{x \in X}, (F_{e,s})_{e \in E_X}),$$

*a  $(\mathcal{V}_{\text{sb}}, \mathcal{R}_{\text{sb}})$ -valued graph function on  $X$ . Suppose there exists a constant  $C_F$  and an  $e' \in E_X$  such that*

$$\begin{aligned} \|F_{e,s}(r)\| &\leq C_F(|r|^{-1} + 1), \quad \forall r > 0, \quad e \in E_X \setminus \{e'\}, \\ \|F_{e',s}(r)\| &\leq C_F(|r|^{-1} + 1)^2, \quad \forall r > 0. \end{aligned}$$

*Suppose that for every  $r > 0$  and  $e \in E_X$  the function  $s \mapsto F_{e,s}(r)$  is continuous. Then for any linked pair partition  $P$  of  $X$  the function  $s \mapsto \mathcal{C}_P(\phi_s)(r)$  is continuous for each  $r \geq 0$  and*

$$\|\mathcal{C}_P(\phi)(r)\| \leq C_F^{|X|} C_1^{|X|-1},$$

*where  $C_1$  is defined in Lemma 12.*

*Proof.* The estimate follows analogous as the estimate in the proof of Lemma 12. The statement about the continuity follows from dominated convergence.  $\square$

*Proof of Theorem 4.* From the proof of Theorem 1 we know various properties about  $C_n$ ,  $G_n$ , and  $T_n$ , and respectively  $\hat{C}_n$  and  $\hat{G}_n$  and  $\hat{T}_n$ . We make the following hypothesis:

$J_n$ : For  $m_1, m_2 \in N_n$  the function  $C_{-m_1, m_2}(r, \eta)$  is continuous and uniformly bounded on  $[0, \infty) \times (0, 1]$  and extends to a continuous function on  $[0, \infty) \times [0, 1]$ .

$J_1$  holds, since  $C_{-1,1} = C_2$ .

Next we show the induction step  $n \rightarrow n+1$ . Thus suppose that  $J_n$  holds. For all  $m_1, m_2 \in N_n$  it follows from the definition that the function  $G_{-m_1, m_2}$  is a continuous uniformly bounded function on  $[0, \infty) \times (0, 1]$  and extends to a continuous functions on  $[0, \infty) \times [0, 1]$ . Let  $m_1, m_2 \in N_n$ . Eq. (5.6) in Theorem 5 implies that  $T_{m_1, m_2}(r, \eta)$  is a continuous function on  $(0, \infty) \times [0, 1]$  and satisfies the following bounds. (Note that there is either at most one  $\tilde{G}_{p,q}$  with a  $0 \in [\min p, \max q]$  or at most one  $\tilde{R}_{\{p,q\}}^\parallel$  with  $0 \in [\min p, \max q]$ .) There exists a constant  $c_n$  such that for all  $r > 0$ ,  $\eta \geq 0$  we have

$$\|P_{\text{at}} T_{-m_1, m_2}(r, \eta) P_{\text{at}}\| \leq c_n, \tag{5.8}$$

$$\|\bar{P}_{\text{at}} T_{-m_1, m_2}(r, \eta) P_{\text{at}}\| \leq c_n(r + \eta)^{-1}, \tag{5.9}$$

$$\|P_{\text{at}} T_{-m_1, m_2}(r, \eta) \bar{P}_{\text{at}}\| \leq c_n(r + \eta)^{-1}, \tag{5.10}$$

$$\|\bar{P}_{\text{at}} T_{-m_1, m_2}(r, \eta) \bar{P}_{\text{at}}\| \leq c_n(r + \eta)^{-2}, \tag{5.11}$$

where we made use of the estimates in the proof of Theorem 1. Using the bounds (5.8)–(5.11) we see that for  $m_1, m_2 \in N_n$  there exists a constant  $C$  such that

$$\|R(r, \eta) T_{-m_1, m_2}(r, \eta) R(r, \eta)\| \leq \frac{C}{(r + \eta)^2}.$$

We conclude from Lemma 15 that  $J_{n+1}$  holds.

Knowing that  $J_n$  holds the definition given in (5.5) implies that  $G_{-m_1, m_2}$  has a continuous extension to  $[0, \infty) \times [0, 1]$ . By Lemma 14 and (5.6) we see that

$$\langle \psi_m(\eta), \psi_m(\eta) \rangle = \langle \varphi_{\text{at}}, T_{-m, m}(0, \eta) \varphi_{\text{at}} \rangle = \langle \varphi_{\text{at}}, G_{-m, m}(0, \eta) \varphi_{\text{at}} \rangle,$$

for which the limit  $\eta \downarrow 0$  exists (observe that the second term in (5.6) does not contribute, since  $R$  contains the projection onto the complement of the unperturbed ground state).

Finally, we will show that the convergence of  $\psi_m(\eta)$ , as  $\eta \downarrow 0$ , follows from dominated convergence. To this end we normal order the creation and annihilation operators and obtain

$$\psi_m(\eta) = \sum_{l=0}^m \psi_{m, l}(\eta),$$

where  $\psi_{m, l}(\eta)$  is an element of  $\mathfrak{h}^{\otimes s^l}$ . Thus the term indexed by  $l$  contains  $l$  creation operators, which originate from positions in the set  $X$ , whereas the other operators on the vertices are contracted. Explicitly, we obtain using the pull through formula and algebraic identities as in the proof of Proposition 1,

$$\begin{aligned} & \psi_{m, l}(\eta)(p_1, p_2, \dots, p_l) \\ &= c_l \sum_{\substack{X \subset N_m \\ |X|=l}} \sum_{\pi: N_l \rightarrow X} \sum_{Y \subset N_m \setminus X} \sum_{\substack{P \text{ pairing of } Y: \\ \text{for each } p \in P \\ \text{there exists an } x \in X \text{ and} \\ \text{a linked path from } p \text{ to } \{0, x\} \\ \text{in } P \cup \{\{0, x\}\}}} \\ & \prod_{y \in Y} \left\{ \int dk_y \right\} \delta_P(k) \tilde{F}_{\{0, \min Z\}}(|K_{\{0, \min Z\}}|_{P \cup P_X}, \eta) \\ & \times \prod_{j \in Z \setminus \max Z} \left\{ I_{j, P \cup P_X}(k_j) \tilde{F}_{\{j, j'_Z\}}(|K_{\{j, r_Z(j)\}}|_{P \cup P_X}, \eta) \right\} \\ & \times I_{\max Z, P \cup P_X}(k_{\max Z}) \hat{F}_{\max Z}(r, \eta) \psi_0|_{\{k_{\pi(s)}=p_s: s \in N_l\}}, \end{aligned} \quad (5.12)$$

where  $c_l$  is a combinatorial factor, we have set  $Z := Y \cup X$  and  $P_X := \{\{0, x\} : x \in X\}$ , and we have used the notations introduced in the definition of the contraction

$$\begin{aligned} G_{j, P}^\# &:= \begin{cases} G_j^* & , \exists p \in P, j = \min p \\ G_j & , \exists p \in P, j = \max p, \end{cases} \\ |K_e|_P &:= \sum_{\substack{p \in P \\ \max e \leq \max p \\ \min p \leq \min e}} |k_{\max p}|, \end{aligned}$$

and we have set

$$\tilde{F}_{\{i, j\}}(r, \eta) := \begin{cases} R(r, \eta), & \text{if } j - i = 1, \\ R(r, \eta) \hat{T}_{j-i}(r, \eta) R(r, \eta), & \text{otherwise,} \end{cases}$$



$$\widehat{F}_{\max Z}(r) := \begin{cases} \mathbf{1}_{\mathcal{H}}, & \text{if } \max Z = m, \\ R(r)\widehat{T}_{m-\max Z}(r), & \text{otherwise.} \end{cases}$$

Now observe that the integrand on the right hand side of (5.12) exists for  $\eta = 0$ , this follows from the pull-through formula and that  $k_j = 0$  is a set of measure zero. A singularity in a possible factor on the very right vanishes because of the projection onto the orthogonal complement of the unperturbed ground state. From the estimate in the proof of Lemma 12 we see that  $\psi_{m,l}(0)$  is square integrable. Furthermore, using the continuity of  $T_{m,n}(r, \eta)$  on  $(0, \infty) \times [0, 1]$  and again the estimate in the proof of Lemma 12 we see from dominated convergence that  $\psi_{m,l}(\eta) \rightarrow \psi_{m,l}(0)$  for  $\eta \downarrow 0$ .  $\square$

## 6 Proof of Theorem 2

In this section we give a proof of Theorem 2. First we show that the ground state and the ground state energy are continuous functions of the coupling constant, that is we verify Hypothesis 2. We recall the notation  $\psi_0 = \varphi_{\text{at}} \otimes \Omega$  and  $E_0 = E_{\text{at}}$ .

**Proposition 3.** *Let  $H(\lambda)$  be given as in (2.2) and assume that Hypothesis 1 is satisfied. Then the following holds.*

(a) *If (2.1) holds, then the ground state energy  $E(\lambda)$  satisfies  $E(\lambda) \leq E_0$  and*

$$E(\lambda) - E_0 = O(|\lambda|^2), \quad (\lambda \rightarrow 0).$$

(b) *If (2.3) holds, then the operator  $H(\lambda)$  has an eigenvector  $\psi(\lambda)$  with eigenvalue  $E(\lambda)$  such that*

$$\|\psi(\lambda) - \psi_0\| = O(|\lambda|), \quad (\lambda \rightarrow 0)$$

*and  $\langle \psi_0, \psi(\lambda) \rangle = 1$  for  $\lambda$  in a neighborhood of zero.*

*Proof.* (a). First we show the upper bound

$$E(\lambda) \leq \langle \psi_0, H(\lambda)\psi_0 \rangle = \langle \psi_0, (H_f + H_{\text{at}} + \lambda\phi(G))\psi_0 \rangle = E_{\text{at}} = E_0.$$

To show the lower bound we complete the square

$$\begin{aligned} H(\lambda) &= H_{\text{at}} + H_f + \lambda\phi(G) \\ &= H_{\text{at}} + \int dk |k| \left[ a(k) + \lambda \frac{G(k)}{|k|} \right]^* \left[ a(k) + \lambda \frac{G(k)}{|k|} \right] - |\lambda|^2 \int \frac{G(k)^* G(k)}{|k|} dk \\ &\geq E_{\text{at}} - |\lambda|^2 \int \frac{\|G(k)\|^2}{|k|} dk. \end{aligned}$$

(b) This is a consequence of the following two claims. We write  $\widehat{\psi}(\lambda) := \frac{\psi(\lambda)}{\|\psi(\lambda)\|}$ .

**Claim 1:** We have that  $\|\bar{P}_\Omega \widehat{\psi}(\lambda)\| = O(|\lambda|)$ .

Calculating a commutator we obtain

$$\begin{aligned} H(\lambda)a(k)\psi(\lambda) &= ([H(\lambda), a(k)] + a(k)H(\lambda))\psi(\lambda) \\ &= (-|k|a(k) - \lambda G(k) + a(k)H(\lambda))\psi(\lambda). \end{aligned}$$

Solving for  $a(k)\psi(\lambda)$  we find

$$(H(\lambda) - E(\lambda) + |k|)a(k)\psi(\lambda) = -\lambda G(k)\psi(\lambda),$$

and by inversion we find for  $k \neq 0$  that

$$a(k)\psi(\lambda) = -\lambda \frac{|k|}{H(\lambda) - E(\lambda) + |k|} \frac{G(k)}{|k|} \psi(\lambda).$$

Thus we obtain for the number operator  $N$  the expectation

$$\begin{aligned} \langle \psi(\lambda), N\psi(\lambda) \rangle &= \int dk \|a(k)\psi(\lambda)\|^2 = |\lambda|^2 \int dk \left\| \frac{|k|}{H(\lambda) - E(\lambda) + |k|} \frac{G(k)}{|k|} \psi(\lambda) \right\|^2 \\ &\leq |\lambda|^2 \int dk \frac{\|G(k)\|^2}{|k|^2} \|\psi(\lambda)\|^2. \end{aligned}$$

Inserting this into the inequality

$$\|\bar{P}_\Omega \psi\|^2 \leq \langle \psi, N\psi \rangle$$

we find that

$$\|\bar{P}_\Omega \hat{\psi}(\lambda)\| = O(\lambda), \quad (\lambda \rightarrow 0).$$

This shows Claim 1.

**Claim 2:** Let  $\bar{P}_{\text{at}} = 1 - P_{\text{at}}$ . Then we have  $\|\bar{P}_{\text{at}} \hat{\psi}(\lambda)\| = O(|\lambda|)$ .

We apply  $\bar{P}_{\text{at}}$  to the eigenvalue equation and obtain

$$\bar{P}_{\text{at}} H(\lambda) \bar{P}_{\text{at}} \psi(\lambda) + \bar{P}_{\text{at}} H(\lambda) P_{\text{at}} \psi(\lambda) = E(\lambda) \bar{P}_{\text{at}} \psi(\lambda).$$

Solving for terms involving  $\bar{P}_{\text{at}} \psi(\lambda)$  we find

$$(\bar{P}_{\text{at}} H(\lambda) \bar{P}_{\text{at}} - E(\lambda) \bar{P}_{\text{at}}) \bar{P}_{\text{at}} \psi(\lambda) = -\bar{P}_{\text{at}} H(\lambda) P_{\text{at}} \psi(\lambda). \quad (6.1)$$

Below we want to show that we can invert the operator on the left and, moreover, we want to estimate the inverse. To this end we will use a Neumann expansion. Let  $\epsilon_1 := \inf \sigma(H_{\text{at}}|_{\text{Ran} \bar{P}_{\text{at}}})$ . By (a) we have in the sense of operators on the range of  $\bar{P}_{\text{at}}$  that

$$(H(0) - E(\lambda)) \bar{P}_{\text{at}} \geq (H(0) - E_0) \bar{P}_{\text{at}} = (H_{\text{at}} + H_f - E_0) \bar{P}_{\text{at}} \geq (\epsilon_1 - E_0) \bar{P}_{\text{at}}.$$

Thus  $(H(0) - E(\lambda)) \bar{P}_{\text{at}}$  is invertible as an operator in  $\text{Ran} \bar{P}_{\text{at}}$ . We note the standard estimates

$$\begin{aligned} \|a(G)\psi\| &\leq \left( \int \frac{\|G(k)\|^2}{|k|} dk \right)^{1/2} \|H_f^{1/2} \psi\| \\ \|a^*(G)\psi\|^2 &\leq \int \|G(k)\|^2 dk \|\psi\|^2 + \int \frac{\|G(k)\|^2}{|k|} dk \|H_f^{1/2} \psi\|^2, \end{aligned}$$

which imply that

$$\|(H_f + 1)^{-1/2}\phi(G)\| = \|\phi(G)(H_f + 1)^{-1/2}\| < \infty.$$

By (a) we find that

$$\begin{aligned} \|(\bar{P}_{\text{at}}(H(0) - E(\lambda))\bar{P}_{\text{at}})^{-1}\bar{P}_{\text{at}}\phi(G)\| &\leq \|(\bar{P}_{\text{at}}(H(0) - E(\lambda))\bar{P}_{\text{at}})^{-1}(H_f + 1)^{1/2}\| \\ &\quad \times \|(H_f + 1)^{-1/2}\phi(G)\| \\ &\leq \sup_{r \geq 0} \left| \frac{(r + 1)^{1/2}}{r + \epsilon_1 - E_0} \right| \|(H_f + 1)^{-1/2}\phi(G)\| =: C_G. \end{aligned} \quad (6.2)$$

By Neumanns Theorem it follows from (6.2) that  $\bar{P}_{\text{at}}(H(\lambda) - E(\lambda))\bar{P}_{\text{at}}$  is invertible on  $\text{Ran}\bar{P}_{\text{at}}$ , if  $|\lambda| < C_G^{-1}$ , and

$$\begin{aligned} &(\bar{P}_{\text{at}}(H(\lambda) - E(\lambda))\bar{P}_{\text{at}})^{-1} \\ &= \sum_{n=0}^{\infty} [-(\bar{P}_{\text{at}}(H(0) - E(\lambda))\bar{P}_{\text{at}})^{-1}\lambda\phi(G)]^n (\bar{P}_{\text{at}}(H(0) - E(\lambda))\bar{P}_{\text{at}})^{-1}. \end{aligned} \quad (6.3)$$

Inserting (6.3) into (6.1) and using again (6.2) we find

$$\begin{aligned} \|\bar{P}_{\text{at}}\hat{\psi}(\lambda)\| &= \|[\bar{P}_{\text{at}}(H(\lambda) - E(\lambda))\bar{P}_{\text{at}}]^{-1}\bar{P}_{\text{at}}H(\lambda)P_{\text{at}}\hat{\psi}(\lambda)\| \\ &\leq \frac{|\lambda|C_G}{1 - |\lambda|C_G} \|P_{\text{at}}\hat{\psi}(\lambda)\|. \end{aligned}$$

This shows Claim 2.

(b) now follows from Claims 1 and 2 by writing

$$\hat{\psi}(\lambda) - \psi_0\langle\psi_0, \hat{\psi}(\lambda)\rangle = \hat{\psi}(\lambda) - P_{\Omega} \otimes P_{\text{at}}\hat{\psi}(\lambda) = \bar{P}_{\Omega}\hat{\psi}(\lambda) + P_{\Omega} \otimes \bar{P}_{\text{at}}\hat{\psi}(\lambda) \rightarrow 0,$$

where the first term on the right hand side tends to zero because of Claim 1 and the second term because of Claim 2. Now  $\psi(\lambda) = \hat{\psi}(\lambda)\langle\psi_0, \hat{\psi}(\lambda)\rangle^{-1}$  is well defined for  $\lambda$  sufficiently close to zero and satisfies (b).  $\square$

*Proof of Theorem 2.* First we show using Theorems 1 and 4 that

$$H_0\psi_{n+1}(0) + V\psi_n(0) = \sum_{k=0}^{n+1} E_k\psi_{n+1-k}(0). \quad (6.4)$$

From the convergence of  $\psi_n(\eta)$  as  $\eta \downarrow 0$  we obtain from the definition of  $E_n$  that

$$E_n = \langle V\psi_0, \psi_n(0) \rangle = \lim_{\eta \downarrow 0} \langle V\psi_0, \psi_n(\eta) \rangle. \quad (6.5)$$

From the definition of  $\psi_n(\eta)$  (compare (3.8)) we see that

$$(H_0 - E_0 + \eta)\psi_{n+1}(\eta) = \bar{P}_0 \left( \sum_{k=1}^{n+1} E_k\psi_{n+1-k}(\eta) - V\psi_n(\eta) \right) \quad (6.6)$$

We claim that the limit  $\eta \downarrow 0$  yields

$$(H_0 - E_0)\psi_{n+1}(0) = \bar{P}_0 \left( \sum_{k=1}^{n+1} E_k \psi_{n+1-k}(0) - V\psi_n(0) \right) \quad (6.7)$$

This clearly holds for  $n = 0$ . Suppose that it holds for all  $m \leq n - 1$ . Then for  $n$  the right hand side of (6.6) converges to the right hand side of (6.7). Since  $H_0$  is a closed operator it follows that the left hand side of (6.6) converges to the left hand side of (6.7). Now (6.7) and (6.5) imply (6.4). By Proposition 3 and (6.4) the assumptions of Lemma 1 are satisfied. Hence Theorem 2 now follows from Lemma 1.  $\square$

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